JORDAN CANONICAL FORM AND ITS APPLICATIONS

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ABSTRACT

This paper gives a basic notion to the Jordan canonical form and its applications. It presents Jordan canonical form from linear algebra, as an "almost" diagonal matrix, and compares it to other matrix factorizations. It also shows that any square matrix is similar to a matrix in Jordan canonical form. Some background on the Jordan canonical form is given.

Keywords : basis,eigen Vectors, eigen values, Jordan chains,nilpotent.

I. INTRODUCTION

The Jordan Canonical Form (JCF) is undoubtably the most useful representation for illuminating the structure of a single linear transformation acting on a finite-dimensional vector space over C (or a general algebraically closed field.

The Jordan Form

Let T $\in$ L(V) and let λ be an eigenvalue of T. For a positive integer r, the subspace $E_r(\lambda) = \ker (T-\lambda I)^r$ is called the generalized eigen space of order r associated with λ. $E_1(\lambda)$ is an eigenspace associated with λ, since V is finite dimensional, there is a positive integer p such that $\{0\} = E_0(\lambda) \subseteq E_1(\lambda) \subseteq \ldots \subseteq E_p(\lambda) = E_{p+1}(\lambda) = \ldots\ldots$

An element $x \in E_r(\lambda) \setminus E_{r-1}(\lambda)$ is called a generalized eigenvector of T of order r corresponding to λ. Clearly if x is a generalized eigenvector of order r then $(T-\lambda I)x$ is a generalized eigenvector of order r-1.

A sequence of non zero vectors $x_1, \ldots, x_k$ is called a Jordan chain of length k associated with eigenvalue λ if

$$T x_1 = \lambda x_1,$$
$$T x_2 = \lambda x_2 + x_1$$
$$\ldots$$
$$T x_k = \lambda x_k + x_{k-1}.$$

A Jordan chain consists of linearly independent vectors.

Proof: Let $x_1, \ldots, x_k$ be a Jordan chain for T associated with eigenvalue λ. Assume that $\alpha_1 x_1 + \ldots + \alpha_k x_k = 0$ and that r is the largest index such that $\alpha_r \neq 0$. Clearly r > 1. Write then $x_r = \sum_{i=1}^{r-1} (\alpha_r^{-1} \alpha_i) x_i$ and operate $(T-\lambda I)^{r-1}$ on both sides to get $x_1 = 0$, a contradiction.

The length of a Jordan chain cannot exceed the dimension of the space and the subspace generated by a Jordan chain is T- invariant. If $B=\{x_1, \ldots, x_k\}$ consists of a Jordan chain, W= < B > and T ' is the Linear operator on W induced by T then

$$[T']_B = \begin{bmatrix} \lambda & 1 & & & \\ & \cdot & \cdot & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \lambda \end{bmatrix}$$

This matrix is called the Jordan block of size k associated with eigenvalue λ and we denote it by $J_k(\lambda)$. Note that $J_k(\lambda) - \lambda I_k$ is a nilpotent matrix of order k.

If V has a basis which is disjoint union of Jordan chains for T, then the matrix representation of T with respect to this basis is a block diagonal matrix with Jordan blocks on the diagonal. This basis is called a Jordan basis of V for T, and the corresponding matrix representation a Jordan canonical form for T.

Existence of Jordan canonical form: If the characteristic polynomial of T splits over K, then V has a Jordan basis for T.

Proof: First assume that T is nilpotent. We prove by induction on n. If T = 0 or in particular n=1, then any basis of V is a Jordan basis. Suppose that $T \neq 0$ and the statement holds for all vector spaces over K of dimension less than n.

Since T is a nilpotent, W = im T is a proper T-invariant subspace of V. Let T ' be the restriction of T on W. Then $T' \in L(W)$ and dim W< n, by induction hypothesis W has a Jordan basis B' for T '. Let $B' = \cup_{i=1}^{k} B'i$, a disjoint union of Jordan chains, that is $B'_i = \{x_{i1},\ldots,x_{in_i}\}$ and $T' x_{i1} = T x_{i1} = 0$ and $T' x_{ij} = T x_{ij} = x_{i\,j-1}$ for $j \in 2(1)n_i$, $i \in 1(1)k$.

Now $x_{11},\ldots, x_{k1}$ a re linearly independent vectors of ker T. Extend it to form a basis of ker T:$\{x_{11}, \ldots, x_{k1}, y_1, \ldots, y_q\}$, $q \geq 0$. Next each $x_{in_k} \in W$, choose $x_{in_k+1} \in V$ such that $T x_{in_k+1} = x_{in_k}$. Now write $B = \cup_{i=1}^{k+q} Bi$, where $B_i = B'_i \cup \{x_i n_{i+1}\}$ for $i \in 1(1)k$, and $B_{k+i} = \{y_i\}$, for $i \in 1(1)q$. We now show that B is a basis of V.

Clearly $|B| = |B'| + k + q$ = dim kerT+ dim imT = dimV = n. Next if

$$\sum_{i=1}^{k}\sum_{j=1}^{n_i+1} \alpha_{ij}\, x_{ij} + \sum_{r=1}^{q} \beta_r\, y_r = 0$$

Where $\alpha_{ij} \in K$ and $\beta_r \in K$. Then operating T on both sides we have

$\sum_{i=1}^{k}\sum_{j=2}^{n_i+1} \alpha_{ij}\, x_{ij-1} = 0$, and so $\alpha_{ij} = 0$ for $j \in 2(1)\, n_i + 1$, $i \in 1(1)k$.

Thus $\sum_{i=1}^{k} \alpha_{i1}\, x_{i1} + \sum_{r=1}^{q} \beta_r\, y_r = 0$ andv which implies that $\alpha_{i1} = 0$ for $i \in 1(1)k$, and $\beta_r = 0$ for $r \in 1(1)q$ as $\{x_{11}, \ldots, x_{k1}, y_1, \ldots, y_q\}$ is a basis for ker T.

Finally if T is an arbitrary then it follows that the minimal polynomial of T is of the form $(x-\lambda_1)^{m1} \ldots (x-\lambda_k)^{mk}$, where $\lambda_1, \ldots, \lambda_k$ are distinct eigen values of T. By primary decomposition theorem $V = V_1 \oplus \ldots \oplus V_k$ where $V_i = \ker (T-\lambda_i)^{mi}$ a T- invariant subspace. Let T_i be the restriction on V_i. Then $T_i \in L(V_i)$ and $S_i = T_i - \lambda_i I$

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is a nilpotent operator on V_i. Therefore V_i has a Jordan basis B_i for S_i and hence for T_i associated with eigenvalue λ_i. Hence $B \cup_{i=1}^{k} B_i$ is a Jordan basis for T.

Let B be a Jordan basis of V for T. Then the number of generalized eigen vectors of T corresponding to eigenvalue λ and of order up to s is dim ker $(T-\lambda I)^s$

Proof. Let B = $\cup_{i=1}^{r} B_i$ be a Jordan basis and is union of disjoint Jordan chains: B_i: $\{x_{i1}, x_{i2}, \ldots, x_{in_i}\}$, i∈ 1 (1)r. Let B_1 B_2,...B_d be all the Jordan chains corresponding to λ. For a positive integer p define

$$\hat{x}_{ip} = \begin{cases} x_{ip} & if\ p \le m_i \\ 0 & if\ p > m_i \end{cases}$$

We prove by induction on s that $\ker(T-\lambda I)^s$ has a basis consisting of nonzero elements of the set { $\hat{x}_{ij}$: i∈ 1(1) l, j ∈ 1(1)s }.This will clearly prove the statement. Since$\{x_{i1}, x_{i2}, \ldots, x_{li}\}$ is a linearly independent subset of ker(T-λI). Thus to prove the induction hypothesis for s = 1 we need to check that this set is actually a basis of ker(T - λI). If v= $\sum_{i=1,j=1}^{r,m_r} \alpha_{ij} x_{ij}$ ∈ ker(T - λI),then

0= (T - λI)v

$$= \sum_{i=1}^{l} \sum_{j=2}^{m_i} \alpha_{ij} x_{ij-1} + \sum_{i=l+1}^{r} \sum_{j=1}^{m_i} \alpha_{ij} (\lambda_i - \lambda) x_{ij} + \sum_{i=l+1}^{r} \sum_{j=2}^{m_i} \alpha_{ij} x_{ij-1}$$

$$= \sum_{i=1}^{r} \sum_{j=1}^{m_i-1} \alpha_{ij+1} x_{ij} + \sum_{i=l+1}^{r} \left(\sum_{j=1}^{m_i-1} (\alpha_{ij}(\lambda_i - \lambda) + \alpha_{ij+1}) + (\lambda_i - \lambda)\alpha_{im1} x_{im1}\right)$$

Therefore α_{ij}=0 for j∈ 2(1)m_i , i∈1(1)l, and for j∈ 1(1)m_i ,i ∈ l+1(1)r. Hence v=$\sum_{i=1}^{l} \alpha_{i1} x_{i1}$ and $\{x_{11}, x_{21}, \ldots, x_{l1}\}$ is a basis of ker(T-λI).

Now assume for s. We now show that $\ker(T-\lambda I)^{s+1}$ has a basis consisting of nonzero elements of { $\hat{x}_{in}$: i∈1(1)l,j∈1(1)s+1}. Since the nonzero elements of this set are already linearly independent, we only need to verify that this set spans $\ker(T-\lambda I)^{s+1}$.

Let v= $\sum_{i=1,j=1}^{r,m_r} \alpha_{ij} x_{ij}$ ∈ $\ker(T - \lambda I)^{s+1}$.Then (T-λI)v ∈ $\ker(T-\lambda I)^s$ = < $\hat{x}_{ij}$: i∈1(1)l, j∈ 1(1)s >

Since

$$(T-\lambda I)v = \sum_{i=1}^{l} \sum_{j=1}^{m_i-1} \alpha_{ij+1} x_{ij} + \sum_{i=l+1}^{r} \left(\sum_{j=1}^{m_i-1} (\alpha_{ij}(\lambda_i - \lambda) + \alpha_{ij+1}) + (\lambda_i - \lambda)\alpha_{im1} x_{im1}\right),$$

We have α_{ij}=0 for j∈ 1(1)m_i ,i ∈l+1(1)r and also that α_{ij}=0 for j∈ s+2(1)m_i , i∈1(1)l,whenever m_i> s+1. Hence

$$v = \sum_{i=1}^{l} \sum_{j=1}^{s+1} \alpha_{ij}\, \hat{x}_{ij}.$$

The number of Jordan chains for T of length m associated with eigen value λ is

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2 dim ker$(T-\lambda I)^m$ - dim ker$(T-\lambda I)^{m+1}$ - dim ker$(T-\lambda I)^{m-1}$,

or

rank ker$(T-\lambda I)^{m+1}$ + rank ker$(T-\lambda I)^{m-1}$ - 2rank ker$(T-\lambda I)^m$

Proof. The number of Jordan chains for T of length at least M associated with λ is exactly the number of generalized eigenvectors of T of order m corresponding to λ which appear in a Jordan basis. By above result this is equal to l_m= dim ker$(T-\lambda I)^m$ - dim ker$(T-\lambda I)^{m-1}$.Therefore the number of Jordan chains for T of length exactly equal to m associated with λ is l_m- l_{m+1}.let T be a linear operator on V and let $c_T(x)$: $(x-\lambda_1)^{n_1}$………$(x-\lambda_k)^{n_k}$ where λ_1,…….. λ_k

are distinct eigenvalues of T. Then the matrix representation of T with respect to jordan basis is : J= diag(J_1,…,J_k), where for each i ∈1(1)k,J_i =diag ($J_{m(i,1)}(\lambda_i)$,………. $J_{m(i,r_i)}(\lambda_i)$).We order the size of these Jordan blocks such that m(i,1)≥ …..≥m(i,r_i).Such a matrix J is the Jordan canonical form or simply the Jordan form of T. Note that for each eigen value λ_i the number r_i and m(i,1), …..,m(i,r_i) are uniquely determined by T. For each eigenvalue λ_i the numbers r_i is the geometric multiplicity of λ_i and m(i,1)+ …..+m(i,r_i)= n the algebraic multiplicity of λ_i. Also it is easy to verify that each J_i is such that $J_i - \lambda_i I_{ni}$, is a nilpotent of order $m_{(i,1)}$Hence the minimal polynomial of T is $(x-\lambda_1)^{m(11)}$………$(x-\lambda_k)^{m(k1)}$.

Now we show by direct matrix multiplications that how an nxn matrix can be transformed to its Jordan form'.

Let J = J_n (0). Then J^t J = I - E_{11}.

Proof: J = E_{12} + …….+ E_{n-1n}. J^t = E_{21} + …….+ $E_{n\,n-1}$.

Thus J^t J = (E_{21} + …….+ $E_{n\,n-1}$)(E_{12} + …….+ E_{n-1n}) = E_{22} + …….+ $E_{n\,n}$ = I - E_{11}.

Let J_m (0) = J, α ∈ C^n and B ∈ $C^{n\,x.n}$. Then

$$\begin{bmatrix} I_m & e_{i+1}\,\alpha^t \\ 0 & I_n \end{bmatrix} \begin{bmatrix} J & e_i\,\alpha^t \\ 0 & B \end{bmatrix} \begin{bmatrix} I_m & -e_{i+1}\,\alpha^t \\ 0 & I_n \end{bmatrix} = \begin{bmatrix} J & e_{i+1}\,\alpha^t B \\ 0 & B \end{bmatrix}$$

Proof: Direct multiplication of the matrices on the left side and using that $Je_{i+1} = e_i$, gives the right side.

Let A be a strictly upper triangular n x n matrix. Then there exists an invertible matrix P and positive integers n_1 , …. , n_k , $n_1 \geq \cdots \geq n_k$ >0 and n_1 +…. +n_k = n such that $P^{-1}AP$ = diag ($j_{n1}(0)$, ….. , $J_{nk}(0)$). Moreover if A has real entries then P will also have real entries. (without proof)

8. **Jordan decomposition theorem**: **A ∈ C^{nxn} is similar to the matrix diag ($J_{n_1}(\lambda_1)$,………. $J_{n_k}(\lambda_k)$),where $J_{ni}(\lambda_i)$ ∈ $C^{n_i x n_i}$ and n_1+……+n_k=n. each $J_{ni}(\lambda_i)$- $\lambda_i I_{ni}$ is of the form of the above result. This form is essentially unique,that its depends only on A and the order of occurrence of eigenvalues.**

Proof. Only uniqueness is required to prove now. For that note that if λ≠0, rank j_m^n =m for all the positive integers n. For j_m(0),rank $j_m(0)^n$=0,if n ≥ m and rank $j(0)_m^{n-1}$ - rank $j(0)_n^m$ =1 for n≤m. write $r_n(\lambda)$=

rank($j_m(\lambda) - \lambda I_m)^n$ .Then $r_{n-1}(\lambda_i)$- $r_n(\lambda_i)$ is the number of Jordan blocks of size ateleast n appearing in J.Therefore the number of Jordan blocks of size exactly equal to n is $(r_{n-1}(\lambda_i) - r_n(\lambda_i)) - (r_n(\lambda_i) - r_{n+1}(\lambda_i)) = r_{n-1}(\lambda_i) - 2r_n(\lambda_i) + r_{n+1}(\lambda_i)$.

Thus two nxn matrices A and B are similar if and only if they have the same eigen characteristics polynomial and for each eigen value λ and positive integer k,

rank $(A-\lambda I)^k$= rank $(B-\lambda I)^k$

II. APPLICATIONS IN GROUP THEORY

We show that being similar to a Jordan canonical form matrix is a relation that partitions the set of square matrices with complex entries $M^{n\times n}$ (C). Of course because only one equivalence class has the identity matrix, Jordan canonical form does not partition the ring of matrices into subrings. We study the question of forming an algebraic structure from the equivalence classes made by partitioning M^n (C). We give a lemma on partitioning a set into equivalence classes.

Lemma : Let $*$ be an equivalence relation on a set X. Then the collection of equivalence classes C (x) forms a partition of X.

So by Lemma , if we can show that ∼ is an equivalence relation on $M^{n\times n}$ (C), it will show that ∼ partitions that set into subsets all similar to a particular Jordan canonical form matrix (so the Jordan form of the subset of each subset "represents" the members of that subset). Recall from Cantor's diagonal proof that the real numbers are uncountable, so the complex numbers, which contain the real numbers, are also uncountable. But note that for an n × n matrix A = $\bigoplus_{i=1}^{n}\lambda i$, with $\lambda_i \in C$, we can have any eigenvalues for A in the complex numbers. Since we have partitioned $M^{n\times n}$ (C) based on eigenvalues, this gives us uncountably many similarity equivalence classes.

Note that for similarity A ∼ B expresses $A = P^{-1}BP$, for some P. But we can have P = I, and then $P^{-1} = I$.

So we find that ∼ is reflexive, because we have an invertible P such that $A = P^{-1}AP$, as A = A.

If A ∼ B, then $A = P^{-1}BP$. But then $B = P AP^{-1}$, and we can just have $Q = P^{-1}$, which gives us $B = Q^{-1}AQ$, so B ∼ A, so ∼ is symmetric.

If A ∼ B and B ∼ C, then $A = P^{-1}BP$ and $B = Q^{-1}CQ$. Note that $B = P AP^{-1}$, so $P AP^{-1} = Q^{-1}CQ$. Thus $A = P^{-1}Q^{-1}CQP$. Let R = QP. As P and Q are invertible, R, their product is also invertible. So $R^{-1} = P^{-1}Q^{-1}$. Thus $A = R^{-1}CR$, so ∼ is transitive. Thus ∼ is an equivalence relation.

Hence, by the partitioning lemma, ∼ partitions $M^{n\times n}$ (C) into subsets of matrices similar to some matrix in Jordan canonical form (and all similar to each other).

We can show that the product of two arbitrary matrices in a certain equivalence class is not in general in that equivalence class; similarly, the equivalence classes are not themselves closed under addition. As well, multiplication of elements in these classes is not well defined, as in general the product of two different choices of matrices as representatives of their equivalence class are not in the same equivalence class.

For example, let us have the following:

$A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $C = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

Clearly both A and B are similar to each other, and B is the Jordan canonical form representative of their equivalence class. We find that AB = C, and that C is not similar to B^2 . Thus we cannot make any groupoid with matrix multiplication as its operation out of similarity classes, because multiplication of similarity classes is not well defined.

III. CONCLUSION

The Jordan canonical form has many uses in linear and abstract algebra. When it can happen, diagaonalization is quite useful and it is good to know that we can always get something very close to diagonal form with Jordan canonical form. We also found that we can partition the set of square matrices $M^{n\times n}$ (C) into equivalence classes of different Jordan canonical forms, but that there does not seem to be any natural algebraic structure that can be formed by similarity classes.

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