Musical Instruments- A Mathematical Approach

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ABSTRACT

Mathematics is used in every field. Mathematics and music are normally thought of two different categories without actual overlap. However, they are indeed related and we also use numbers and mathematics to describe music. Counting, rhythm, scales, intervals, patterns, symbols, tone pitch. The notation of composers are connected to mathematics of musical instruments, Pythagoreans had discovered the combinations of pitches with simple ratio of frequencies. A number of different tones can be produced on the willow flute. Through this paper, we will put light on the Norwegian folk flute called willow flute, which depends on the mathematics for sound. The solution of the particular wave equation gives the amplitude & pitch of the sound.

I. INTRODUCTION

The history of musical instruments has been started ten thousand years ago. Recently, a 9,000-year-old flute found in China was said to be the world's oldest playable instrument. This history shows that humans have long been related with producing sounds of different frequencies. Also, the finger holes on flutes indicates that the historic musicians had some concept of a musical scale. The study of the mathematics of musical instruments has been started by Pythagoreans, who uses simple ratios of frequencies to find different combinations of pitches of pleasing sound. The problems of tuning and temperament are recent one.

II. THE WILLOW FLUTE

In this paper we will put light on physical properties of a Norwegian folk flute called the willow flute. This flute can be considered as primary instrument as it does not depend on finger holes to produce different pitches by this instrument. Rather, player can select series of pitches by varying the strength with which he blows into the flute. These series of pitches is known as harmonics whose frequencies are integral multiples of the flute's lowest tone, called the fundamental.

The willow flute is a member of recorder family. This flute is made by a hollow willow branch. It's one end is open and the other contains a slot into which the player blows, forcing air through the notch in the flute. The resultant vibration creates standing waves inside the instrument. The frequency of these waves determines the pitch. There are finger holes in the recorder that allow the player to change the frequency of the standing waves, whereas the willow flute has no such finger holes. However, the Tune Willow Dance, performed by Hans Brimi on the willow flute, shows that certain number of different tones can be produced on the willow flute. Now the question arise that how is this possible?

The answer lies in the mathematics of sound waves. Let u denotes the pressure in the tube. x be the position along the length of the tube and t be time. Since the pressure across the tube is almost constant, we can neglect it's direction. The units are chosen in such a way that the pressure outside the tube is zero.

The one-dimensional wave equation $a^2\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u}{\partial t^2}$ gives the behavior of air molecules in the tube, where `a` is a positive constant . Since the tube is open from both ends, so the pressure at the ends is same as the outside pressure.

This implies , if L is the length of the tube , u(0,t)=0 and u(L,t)=0. Solutions to the wave equation are sums of solutions of the form

u(x,t)= $\sin\frac{n\pi x}{L}\left(b\sin\frac{an\pi t}{L}+c\cos\frac{an\pi t}{L}\right)$ where n=1,2,3,…… and b,c are constant.

Now we will show how does our solution predict the possible frequencies of tones produced by the flute. For this, consider solutions which contains one value of n. So fix n , x and vary t. The pressure, u , varies periodically with period 2L/an.

Therefore, frequency =an/2L; n=1,2,3,…..

This suggests that there are two ways to play a wind instrument ;either change the length L or change n. As in slide whistle, by varying L continuously, one can produce continuous changes in pitch. Mostly L is changed by making holes in the tube that gives discrete changes in pitch. Secondly the pitch can be varied by changing n. These different sets of pitches by varying n are the harmonics. The pitch with frequency an/2L is called nth harmonic; if n=1 the pitch is called fundamental.

III. DRUMES AND OTHER HIGHER –DIMENSIONAL INSTRUMENTS

Till now, we have discussed about one-dimensional instruments such as stringed and wind instruments. Percussion instruments such as drums and bells do not. Why is that?

Imagine a drum with a circular drumhead as a circular domain of radius R around the origin in R^2 that obeys the wave equation with fixed boundary. Using polar coordinates (r, $\emptyset$) and separation of variables, we get the transversal displacement Type equation here.of the drumhead at time t, given by

$$f(r,\emptyset,t)=g(t)f_1(r)f_2(\Phi)$$

$$g''(t)+c^2\gamma g(t)=0$$

$$f_2''(\emptyset)+\mu f_2(\emptyset)=0$$

$$f_1''(r)+\frac{1}{r}f_1'(r)+(\gamma-\frac{\mu}{r^2})\,f_2(r)=0$$

where the constant c is related to physical properties of the material, and , γ,μ are determined by the conditions f2(- $\pi)=f2(\pi)$ and f1(R) =0

The solutions to equations (1) can be easily determined. The constraints on force f2 are given by $\mu = m^2$. This implies that equation (2) is the m-th Bessel equation, having solutions given in the form of the m-th Bessel function as f1(r) $=J_{m(R\sqrt{\gamma})}$ Since, the drumhead is fixed along it's boundary.

So, f1(r) $=J_{m(R\sqrt{\gamma})}=0$ Solving this equation , we see that γ can have only the values

$$\gamma_{n}=(\frac{x_n^m}{R})^2$$

Where x_n^m are the zeros of the m^{th} Bessel equation. The different values of γ gives the frequencies of oscillations of different modes as in case of willow flute. Since the zeros are irrationally related, this implies that the frequencies of oscillations of the drumhead cannot be rational multiples of each other. That is the reason, why drums using a freely oscillating circular membrane and one-dimensional instruments produce notes having discernibly different. But as exceptions always exist. There is one instrument which does not fit within this picture, is the tympanum. Although the membrane of tympanum is two-dimensional, even then its sound is similar to one-dimensional instruments. Unlike a tambourine, the tympanum has a closed bottom and its variations change the pressure in the cavity beneath the oscillating membrane.

Therefore, its membrane is not free to oscillate. So additional nonlinear forcing terms have to be added to the wave equation to its behavior accurately.

Also, we are not only to discuss about circular drums. So consider a wave equation on a general domain D in R^2 and take the solutions that satisfy F(x,y,t)=0 on the boundary ∂D . Separating variables as

$f(x,y,t) = \varphi\,(t)\emptyset(x,y)$ where $\nabla^2\,\emptyset + \gamma\emptyset = 0\;\; in\, D$

And $\emptyset = 0\; on\, \partial D$

Also ,we know such equations have solution only for some values of γ known as eigen values. These eigen values are dependent of the shape of drum D.

In the article, Can one hear the shape of a drum? M. Kac asked whether two drums having same eigen values, always have the same shape. This problem remained unsolved for 24 years until Gordon et al proved that two non-congruent drums can have the same eigen values.

Now, we'll try to characterize the sound of three- dimensional instruments. All three- dimensional instruments fall in the class of percussion instruments. Considering the wave equation of some simple geometric shapes- such as, a rode- we can find that the frequencies of the different modes are not rationally related.

However, there exist some three- dimensional instruments having sounds similar to that of one- dimensional instruments such as marimba, claves and others. Some of tree- dimensional objects for examples rods, vibrate predominantly in a one- dimensional manner. Whereas, the bars on a marimba corresponding to lower tones have deep arches on one side. Also, these are cut in such way that first two modes of vibration of the bar are rationally related.

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