Three Species Ecological Model with Harvesting Efforts

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ABSTRACT

The aim of this paper is to study the stability analysis of three species Ecological model consists of a Prey (N_1), a predator (N_2) and a competitor (N_3) .The competitor (N_3) is competing with the Prey Species (N_1) and neutral with the predator (N_2). In addition to that, the death rates and harvesting efforts of all three species are also considered for investigation. The model is characterized by a system of non linear ordinary differential equations. All the eight equilibrium points of the model are identified and local stability is discussed at interior equilibrium point. The global stability is studied by constructing a suitable Lyapunov's function. Further the harvesting effort is studied with different efforts on three species which effects stability analysis using Numerical simulation in aid of Mat lab.

Keywords: Prey, Predator, Competitor, Equilibrium points, local stability, Global stability, Numerical simulation.

I.INTRODUCTION

Lokta [1] and Volterra [2] studied mathematical models in population dynamics. Kapur [3, 4] discussed mathematical models related to biology, ecology, medicine etc. paparao [5, 6] studied the stability analysis three species ecological model with a prey , predator and competitor. Freed man [7] studied different interaction in population dynamics with detailed analysis and Paul Colinvaux [8] studied mathematical aspects of ecological models .The models in Ecology are represented by differential equations. The applications of differential equations are widely studied by Braun [9] and Simon's [10]. In this paper, we study the stability analysis of three species ecological model with a prey, predator and competitor. The model is characterized by system of non linear ordinary differential equations further equilibrium points are derived. Local stability analysis is discussed at the interior equilibrium point. The global stability is studied by constructing a suitable Lyapunov's function. Further studied the harvesting efforts for which the system is asymptotically stable and stable using MAT LAB simulation .

II.MATHEMATICAL MODEL

Consider Mathematical model for the system is

$$\begin{aligned}\frac{dN_1}{dt} &= a_1(1-k_1)N_1 - \alpha_{11}N_1^2 - \alpha_{12}N_1N_2 - \alpha_{13}N_1N_3 - d_1N_1 \\ \frac{dN_2}{dt} &= a_2(1-k_2)N_2 - \alpha_{22}N_2^2 + \alpha_{21}N_1N_2 - d_2N_2 \\ \frac{dN_3}{dt} &= a_3(1-k_3)N_3 - \alpha_{33}N_2^2 - \alpha_{31}N_1N_3 - d_3N_3\end{aligned} \qquad (2.1)$$

2.1: Nomenclature:

S.No	Parameter	Description
1	N_1, N_2 & N_3	Populations of the prey, predator and competitor respectively
2	a_1, a_2, a_3	Natural growth rates of prey, predator and competitor respectively
3	k_1,k_2,k_3	Harvesting efforts of prey, predator and competitor respectively
4	$\alpha_{ii}\,(i = 1, 2, 3)$	Rate of decrease of prey, predator and competitor populations respectively due to inter species competition.
5	α_{12}	Rate of decrease of the prey due to inhibition by the predator
6	α_{21}	Rate of increase of the predator due to successful attacks on the prey
7	α_{13}	Rate of decrease of the prey due to the competition with the competitor
8	α_{31}	Rate of decrease of the competitor due to the competition with the prey
9	d_1,d_2,d_3	Death rates of prey, predator and competitor respectively

Note: $0 < k_i < 1 (i = 1, 2, 3)$ Assume throughout the analysis $(a_i(1-k_i) - d_i) > 0\ (i = 1, 2, 3)$

III.EQUILIBRIUM STATES

The possible equilibrium points are

I.E_1: The extinct state $\overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = 0.$ (3.1)

II.E_2: The state in which only the predator survives and the prey and competitor are extinct.

$$\overline{N}_1 = 0, \overline{N}_2 = \frac{a_2(1-k_2) - d_2}{\alpha_{22}}, \overline{N}_3 = 0. \qquad (3.2)$$

III. E_3: The state in which both the prey and the predators extinct and competitor alone

survive. $\overline{N}_1 = 0, \overline{N}_2 = 0, \overline{N}_3 = \dfrac{a_3(1-k_3)-d_3}{\alpha_{33}}.$

(3.3)

IV. E_4: The state in which both the predator and competitor extinct and prey survive.

$$\overline{N}_1 = \frac{a_1(1-k_1)-d_1}{\alpha_{11}}, \overline{N}_2 = 0, \overline{N}_3 = 0. \qquad (3.4)$$

V. E_5 : The state in which both the prey and the predators exist and competitor extinct

$$\overline{N}_1 = \frac{(a_1(1-k_1)-d_1)\alpha_{22} - (a_2(1-k_2)-d_2)\alpha_{12})}{\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}}, \overline{N}_2 = \frac{(a_2(1-k_2)-d_2)\alpha_{11} + (a_1(1-k_1)-d_1)\alpha_{21})}{\alpha_{11}\alpha_{22}+\alpha_{12}\alpha_{21}}, \overline{N}_3 = 0$$

(3.5)

This case arise only when $\alpha_{22}(a_1(1-k_1)-d_1) > \alpha_{12}(a_2(1-k_2)-d_2))$ (3.5.a)

VI. E_6 : The state in which both prey and competitor exist and predator extinct,

$$\overline{N}_1 = \frac{\alpha_{33}(a_1(1-k_1)-d_1) - \alpha_{13}(a_3(1-k_3)-d_3))}{(\alpha_{11}\alpha_{33}-\alpha_{13}\alpha_{31})}, \overline{N}_2 = 0, \overline{N}_3 = \frac{\alpha_{11}(a_3(1-k_3)-d_3) - \alpha_{31}(a_1(1-k_1)-d_1)}{(\alpha_{11}\alpha_{33}-\alpha_{13}\alpha_{31})}$$

$$\begin{aligned} &\alpha_{11}\alpha_{33} > \alpha_{13}\alpha_{31}, \frac{\alpha_{33}}{\alpha_{13}} > \frac{(a_3(1-k_3)-d_3)}{(a_1(1-k_1)-d_1)} \ \& \ \frac{\alpha_{11}}{\alpha_{31}} > \frac{(a_1(1-k_1)-d_1)}{(a_3(1-k_3)-d_3)} \ or \\ &\alpha_{11}\alpha_{33} < \alpha_{13}\alpha_{31}, \frac{\alpha_{33}}{\alpha_{13}} < \frac{(a_3(1-k_3)-d_3)}{(a_1(1-k_1)-d_1)} \ \& \ \frac{\alpha_{11}}{\alpha_{31}} < \frac{(a_1(1-k_1)-d_1)}{(a_3(1-k_3)-d_3)} \end{aligned} \qquad (3.6)$$

VII. E_7: The state in which both predator and competitor exist and prey extinct,

$$\overline{N}_1 = 0, \overline{N}_2 = \frac{a_2(1-k_2)-d_2}{\alpha_{22}}, \overline{N}_3 = \frac{a_3(1-k_3)-d_3}{\alpha_{33}}. \qquad (3.7)$$

VIII. E_8 : The state in which prey, predator and competitor exist

$$\overline{N}_1 = \frac{[a_1(1-k_1)-d_1]\alpha_{22}\alpha_{33} - [a_2(1-k_2)-d_2]\alpha_{12}\alpha_{33} - [a_3(1-k_3)-d_3]\alpha_{13}\alpha_{22})}{\alpha_{22}(\alpha_{11}\alpha_{33}-\alpha_{13}\alpha_{31})+\alpha_{12}\alpha_{21}\alpha_{33}}.$$

$$\overline{N}_2 = \frac{[a_2(1-k_2)-d_2](\alpha_{11}\alpha_{33}-\alpha_{13}\alpha_{31}) + \alpha_{21}\alpha_{33}[a_1(1-k_1)-d_1] - \alpha_{13}[a_3(1-k_3)-d_3]}{\alpha_{22}(\alpha_{11}\alpha_{33}-\alpha_{13}\alpha_{31})+\alpha_{12}\alpha_{21}\alpha_{33}},$$

$$\overline{N}_3 = \frac{[a_3(1-k_3)-d_3]\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}[a_3(1-k_3)-d_3] + \alpha_{31}[a_2(1-k_2)-d_2] - \alpha_{31}\alpha_{22}[a_1(1-k_1)-d_1]}{\alpha_{11}(\alpha_{22}\alpha_{33}-\alpha_{23}\alpha_{32})+\alpha_{12}\alpha_{21}\alpha_{33}}.$$

(3.8)

The equilibrium state exist only when

$$\overline{N_1} > 0\overline{N_2} > 0, \overline{N_3} > 0. \tag{3.8.a}$$

IV. LOCAL STABILITY OF THE EQUILIBRIUM POINT E_8:

Theorem: The positive equilibrium point $E_8\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$ is locally asymptotically stable

If $\alpha_{11}\alpha_{33} > \alpha_{13}\alpha_{31}$

Proof: The variational matrix is given by

$$J = \begin{bmatrix} a_1(1-k_1) - 2\alpha_{11}\overline{N_1} - \alpha_{12}\overline{N_2} - d_1 & -\alpha_{12}\overline{N_1} & -\alpha_{13}\overline{N_3} \\ \alpha_{21}\overline{N_2} & a_2(1-k_2) - 2\alpha_{22}\overline{N_2} + \alpha_{21}\overline{N_1} - \alpha_{23}\overline{N_3} - d_2 & 0 \\ -\alpha_{31}\overline{N_1} & 0 & a_3(1-k_3) - 2\alpha_{33}\overline{N_3} - \alpha_{32}\overline{N_2} - d_3 \end{bmatrix} \tag{4.1}$$

$$= \begin{bmatrix} -\alpha_{11}\overline{N_1} & -\alpha_{12}\overline{N_1} & -\alpha_{13}\overline{N_3} \\ \alpha_{21}\overline{N_2} & -\alpha_{22}\overline{N_2} & 0 \\ -\alpha_{31}\overline{N_1} & 0 & -\alpha_{33}\overline{N_3} \end{bmatrix} \tag{4.2}$$

The characteristic equation of (4.2) be $\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0$ (4.3)

Where

$$a_1 = \alpha_{11}\bar{N}_1 + \alpha_{22}\bar{N}_2 + \alpha_{33}\bar{N}_3,$$

$$a_2 = \left(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\right)\overline{N_1}\,\overline{N_3} + \left(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}\right)\overline{N_1}\,\overline{N_2} + \left(\alpha_{33}\alpha_{22}\right)\overline{N_2}\,\overline{N_3},$$

$$a_3 = [\left(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\right)\alpha_{22} + \alpha_{21}\alpha_{12}\alpha_{33}]\overline{N_1}\,\overline{N_2}\,\overline{N_3}.$$

Let $D_1 = a_1 = \alpha_{11}\overline{N_1} + \alpha_{22}\overline{N_2} + \alpha_{33}\overline{N_3} > 0$

$$D_2 = a_1 a_2 - a_3 > 0$$

$$D_3 = a_3(a_1a_2 - a_3) > 0 \tag{4..4}$$

Clearly $D_1 = a_1 > 0$, calculate $D_2 = \left(a_1a_2 - a_3\right) > 0$ and $D_3 = a_3(a_1a_2 - a_3) > 0$.

$$D_2 = a_1a_2 - a_3 = \alpha_{11}\left(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\right)\overline{N_1}^2\,\overline{N_3} + \alpha_{33}\left(\alpha_{11}\alpha_{33} - \alpha_{13}\alpha_{31}\right)\overline{N_1}\,\overline{N_3}^2 + \alpha_{11}\alpha_{22}\alpha_{33}\overline{N_1}\,\overline{N_2}\,\overline{N_3}$$
$$+\alpha_{22}\alpha_{33}\left(\alpha_{11}\overline{N_1} + \alpha_{22}\overline{N_2} + \alpha_{33}\overline{N_3}\right) + (\alpha_{11}\overline{N}1 + \alpha_{22}\overline{N_2})\left(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}\right)\overline{N_1}\,\overline{N_2} + \alpha_{11}\alpha_{33}{}^2\,\overline{N_1}\,\overline{N_3}^2$$

$$D_2 > 0 \text{ if } \alpha_{11}\alpha_{33} > \alpha_{13}\alpha_{31} \tag{4.5}$$

And $D_3 = a_3(a_1a_2 - a_3) > 0$ if $\alpha_{11}\alpha_{33} > \alpha_{13}\alpha_{31}$

By Routh-Hurwitz criteria the system is stable since the determinants are positive

Hence the positive equilibrium point $E_8\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$ is locally asymptotically stable if $\alpha_{11}\alpha_{33} > \alpha_{13}\alpha_{31}$

V.GLOBAL STABILITY

Theorem: The positive equilibrium point $E_8\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right)$ is globally asymptotically stable if $\alpha_{12} > \alpha_{21}$

Proof: Let the Lyapunov function be

$$V\left(N_1, N_2, N_3\right) = \left[N_1 - \overline{N_1} - \overline{N_1}\log\left(\frac{N_1}{\overline{N_1}}\right)\right] + l_1\left[N_2 - \overline{N_2} - \overline{N_2}\log\left(\frac{N_2}{\overline{N_2}}\right)\right] + l_2\left[N_3 - \overline{N_3} - \overline{N_3}\log\left(\frac{N_3}{\overline{N_3}}\right)\right] \tag{5.1}$$

Clearly $V\left(\overline{N_1}, \overline{N_2}, \overline{N_3}\right) = 0$ & $V(N_1, N_2, N_3) > 0$

The time derivate of V along the solutions of equations (2.1) is

$$\frac{dV}{dt} = \frac{dN_1}{dt}\left[1 - \frac{\overline{N_1}}{N_1}\right] + l_1\frac{dN_2}{dt}\left[1 - \frac{\overline{N_2}}{N_2}\right] + l_2\frac{dN_3}{dt}\left[1 - \frac{\overline{N_3}}{N_3}\right] \tag{5.2}$$

$$= [N_1 - \overline{N_1}]\left[a_1(1-k_1) - d_1 - \alpha_{11}N_1 - \alpha_{12}N_2 - \alpha_{13}N_3\right] + l_1[N_2 - \overline{N_2}]\left[a_2(1-k_2) - d_2 - \alpha_{22}N_2 + \alpha_{21}N_1\right]$$
$$+ l_2[N_3 - \overline{N_3}]\left[a_3(1-k_3) - d_3 - \alpha_{33}N_3 - \alpha_{31}N_1\right] \tag{5.3}$$

Substitute $a_1(1-k_1)-d_1=\alpha_{11}\overline{N}_1+\alpha_{12}\overline{N}_2++\alpha_{13}\overline{N}_3, a_2(1-k_2)-d_2=-\alpha_{21}\overline{N}_1+\alpha_{22}\overline{N}_2,$
$a_3(1-k_3)-d_3=\alpha_{33}\overline{N}_3+\alpha_{31}\overline{N}_1$

using the relation $ab\le\frac{a^2+b^2}{2}$

We get

$$\frac{dV}{dt}<-\left(\alpha_{11}+\frac{1}{2}[\alpha_{12}-\alpha_{21}]+\frac{1}{2}[\alpha_{13}+\alpha_{31}]\right)\left[N_1-\overline{N_1}\right]^2-\left(\alpha_{22}+\frac{1}{2}[\alpha_{12}-\alpha_{21}]\right)\left[N_2-\overline{N_2}\right]^2$$
$$-\left(\alpha_{33}+\frac{1}{2}[\alpha_{31}+\alpha_{13}]\right)\left[N_3-\overline{N_3}\right]^2<0$$

$\frac{dV}{dt}<0$ if $\alpha_{12}>\alpha_{21}$

Hence positive equilibrium point $E_8\left(\overline{N_1},\overline{N_2},\overline{N_3}\right)$ is globally asymptotically stable if $\alpha_{12}>\alpha_{21}$

VI.NUMERICAL EXAMPLE

Fig A: denotes Time series analysis of prey, predator and competitor populations

Fig B: denotes Phase portrait of prey, predator and competitor populations

Example 1: Let a_1=4, a_2=.63, a_3=2, α_{11}=0.1010, α_{12}=0.4, α_{13}=0.001, α_{22}=0.1010, α_{21}=.1010, α_{33}=0.5010, α_{31}=.6, d_1 = 0.02, d_2 = 0.02, d_3 = 0.03, N_1 = 15; N_2 = 9; N_3 = 20

The system is asymptotically stable and converging to a fixed Equilibrium point E (3, 9, 1) when there is no harvesting effort is induced.

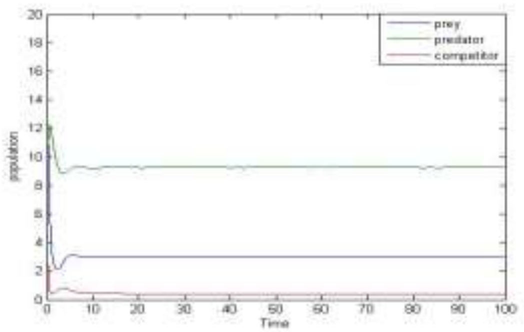


fig1. (A)

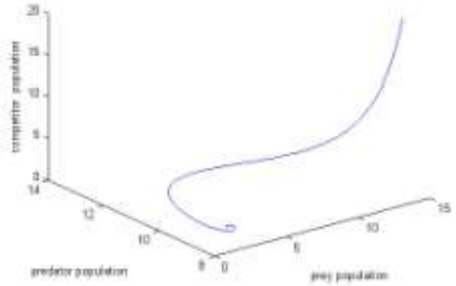


fig 1. (B)

A. For the above set of parametric values with $k_1 = 0.1$, $k_2 = 0.1$, $k_3 = 0.1$, initially the system exhibits oscillatory behavior later on it stabilizes and quenching to the equilibrium point E (3,8,1) shows that the system is asymptotically stable

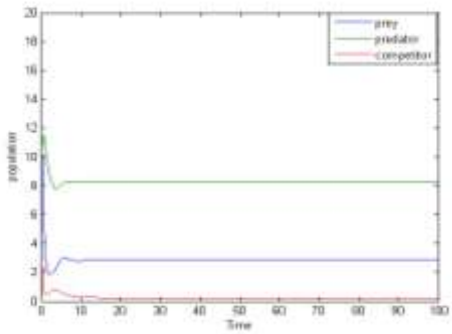


fig2. (A)

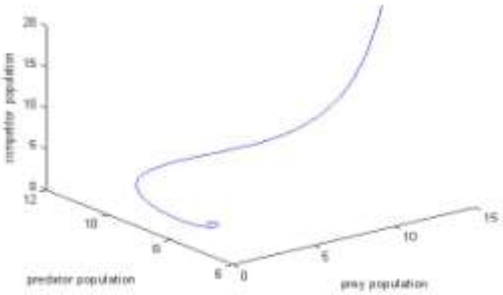


fig 2. (B)

B. For the above set of parametric values with $k_1 = 0.5$, $k_2 = 0.5$, $k_3 = 0.5$, initially the system exhibits oscillatory behavior later on it stabilizes and quenching to the equilibrium point E (2,5,0) shows that the system is asymptotically stable.

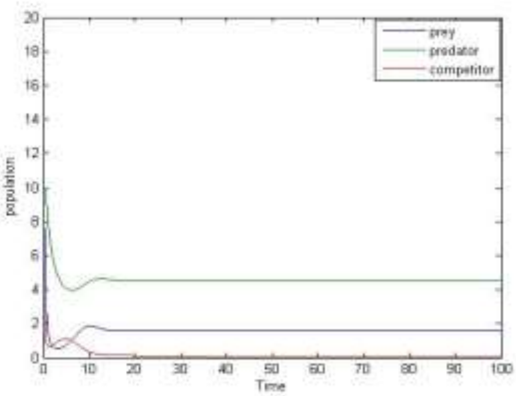


fig3. (A)

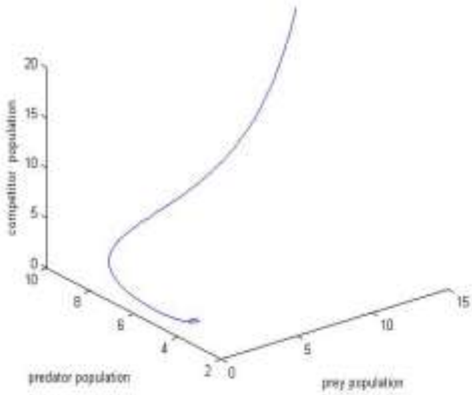
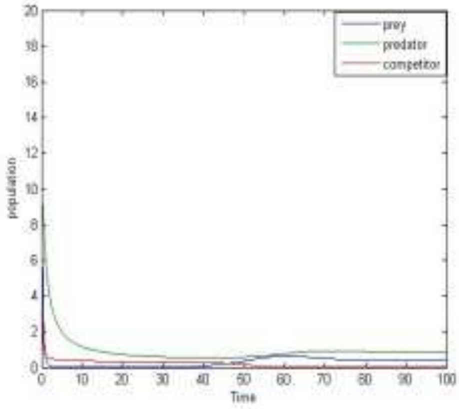


fig 3. (B)

C. For the above set of parametric values with $k_1 = 0.9$, $k_2 = 0.9$, $k_3 = 0.9$, The three populations are almost extinct and the system is stable and converging to origin.



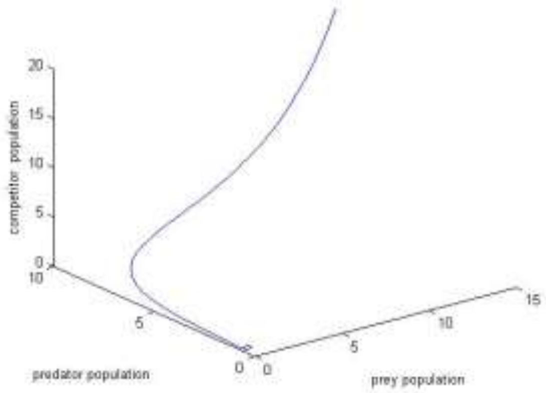
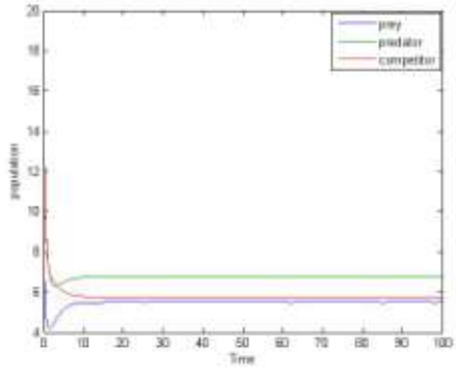


fig4. (A) fig4. (B)

Example 2: Let $a_1=2$; $a_2=1$; $a_3=2$; $\alpha_{11}=0.1$; $\alpha_{12}=0.1$; $\alpha_{13}=0.1$; $\alpha_{22}=0.2$; $\alpha_{21}=.1$; $\alpha_{33}=0.2$; $\alpha_{31}=.1$; $d_1 = 0.02$, $d_2 = 0.02$, $d_3 = 0.02$, $N_1=10$; $N_2 =15$; $N_3=20$.

The system is stable and the populations are converging to fixed point and hence the system is neutrally stable converging to the equilibrium point E (6, 7, 6)



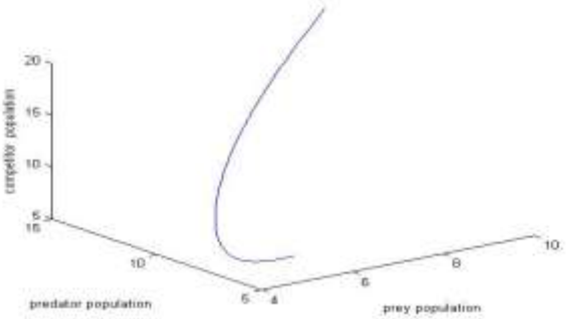


fig 7. (A) fig 7. (B)

For different Harvesting coefficients the plots are shown below

A. For the above set of parametric values with $k_1 = 0.1$, $k_2 = 0.1$, $k_3 = 0.1$, The three populations are stabilizes to a fixed equilibrium point E (5,6, 5) ,hence the system is stable .

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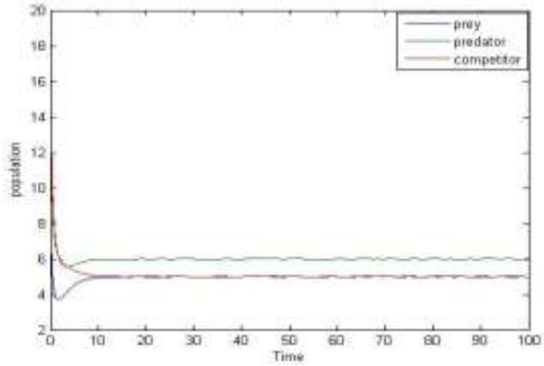


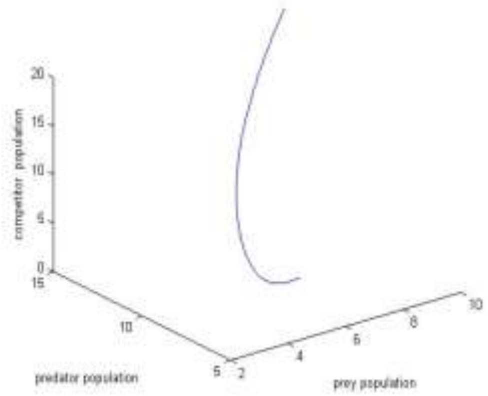
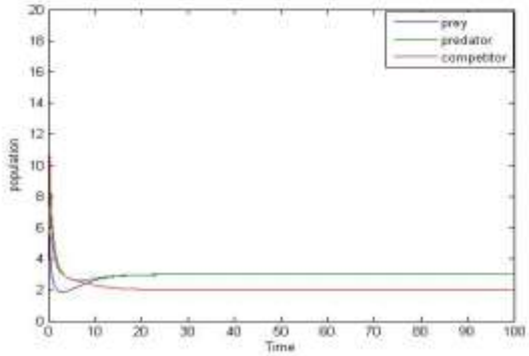


fig 8. (A) fig 8. (B)

B. For the above set of parametric values with $k_1 = 0.5$, $k_2 = 0.5$, $k_3 = 0.5$ the three populations are stabilizes to a fixed equilibrium point E (3, 3, 2), hence the system is asymptotically stable.



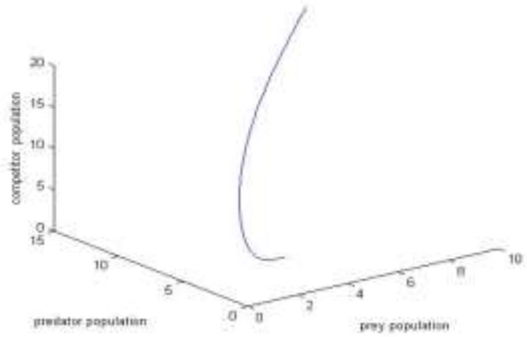
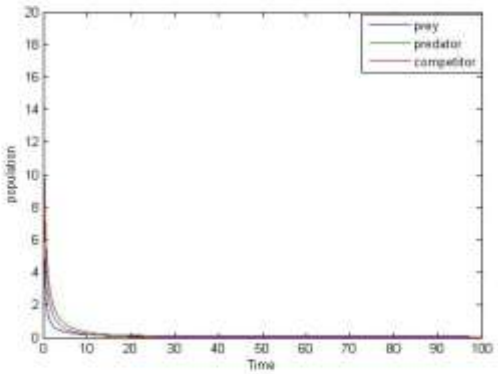


fig 9. (A) fig 9. (B)

C. For the above set of parametric values with $k_1 = 0.9$, $k_2 = 0.9$, $k_3 = 0.9$, The three populations are converging to origin , hence the system is stable



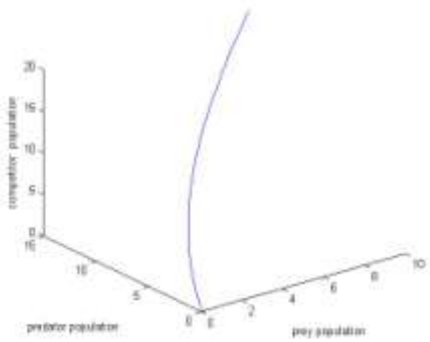


fig 10. (A) fig 10. (B)

VII.CONCLUSION

We study the stability analysis of three species Ecological model consists of a Prey (N_1), predator (N_2) and a competitor (N_3) .The competitor (N_3) is competing with the Prey Species (N_1) and neutral with the predator (N_2). In addition to that, the death rates and harvesting efforts of all three species are also considered for investigation. The model is characterized by a system of non linear ordinary differential equations. All the eight equilibrium points of the model are identified and their local stability is discussed at interior equilibrium point. The global stability is studied by constructing a suitable Lyapunov's function.

The numerical simulation is carried out for different parametric values of the system shown in equation (2.1). The simulation shows that for the specified parameters in examples from 1 and 2 with different harvesting efforts of the three species the system exhibits stable behiviour as on the harvesting parameters are close to the value one the three populations are converging to origin , shows that the system is stable.. Hence harvesting shows a significant effect on the stability of the system.

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