4th International Conference on Science, Technology and Management
India International Centre, New Delhi (ICSTM-16)
15th May 2016, www.conferenceworld.in ISBN: 978-81-932074-8-2

REVIEW PAPER ON FRACTIONAL FOURIER TRANSFORM

Rachit Patel

ECE Department, ABES Institute of Technology, (India)

ABSTRACT

Fractional Fourier Transform (FRFT) is generalization of the Classical Fourier Transform (FT). The FRFT is realized based on a special parameter α, which is known as an angle rotation in time frequency plane. Here α= (aπ)/2, a € R .Here 'a' is equal to the 1 is a special case for Fourier Transform. The whole paper is divided into different sections illustrated as- first there is an introduction to FRFT and its properties including FRFT of some signals, second including how FRFT works in the time-frequency distribution (as a rotationprocess) and then its applications in various domains. Important part is that how it can be used for the purpose of signal security using multiorders phenomenon.

Keywords: Digital signal processing, Fourier Transform, FRFT, Image compression, Orthogonal Frequency Division Multiplixing.

I. INTRODUCTION

The term Fraction power for the Fourier operators appears in mathematical literature proposed by Wiener in 1929. After that it is used in optics, quantum mechanics, signal processing and in communications. After 1990's many publications have been published in this field and it is still going on [1][7]. Fractional Fourier Transform, commonly referred to as FRFT is the generalization of the Fourier Transform (FT). The FRFT was firstly introduced by Victor Nami as in 1980's in signal processing field. Victor Nami as also found that, the other transform can also fractionalize [3]. FRFT is more suitable for the non-stationary signals than FT due to its time-frequency characteristics. Also FRFT signals can be consider as the decompositions of the signals in terms of chirp sets. Chirp sets have the same sweeping rates and different initial frequency. Therefore FRFT used in target detection and parameter estimations [8]. FRFT has been used in several applications such as area in optics[16], image processing[17], solution of differential equations, quantum mechanics, signal detector, correlation and pattern recognition, signal and image recovery, noise removal, OFDM, Image smoothing, encryption and decryption and study of space or time-frequency distributions [3], etc. FRFT can be used in each area where we are applying FT. The purpose of this paper is knowledge about the FRFT and recent development in the time-frequency field. And also knowledge about some others existing fractional transform.

The text organized as follows: The author has divided whole paper in different sections –second section there is a short description of Fractional Fourier Transform i.e. (FRFT), third section includes representation of FRFT in the time-frequency plane along with rotation, fourth section includes properties of FRFT, fifth section includesapplications of FRFT in different domain and last section is the reference part.

II. FRACTIONAL FOURIER TRANSFORM

There are so many definition of the FRFT, but the most intuitive way of defining the FRFT[3] is by generalizing theconcept of rotation over an angle $\pi/2$ in the classical FT condition. The generalization of the FT is known as FRFT in mathematical literature. FRFT realized based on a special parameter α (a rotation angle). Here $\alpha = a\pi/2$, a € R. The value of a can take any value. The period of α is 0 to 2π.

III. REPRESENTATION OF THE FRFT

Whenever we talk about the FT, we go time domain to frequency domain. In this case the angle $\alpha = 90°$ took place as shown in the figure. Now we want to find out the component between the time axis and frequency axis.

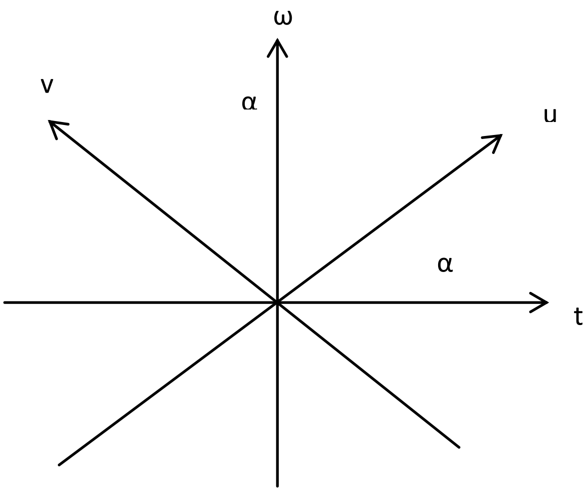

Figure1:-Notaional convention for the variables and rotated version

Iftime and frequency axis (t and ω) rotated by an angle α where $\alpha = a\pi/2$. Where a € R is counter clockwise. Then we represent the rotated variable as u and v in the form of matrix.

Thus

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} t \\ \omega \end{bmatrix}$$

$u$and v are always orthogonal.

Let f (t) any signal in time domain with respect to the horizontal axis and its FT(Ƒf)(t) is a function of variable ω on the vertical axis. Thus by the FT representation, axis move form representation in the time axis to the representation in the ω axis with respect to the counter clockwise rotation over an angle π/2 in the (t, w) plain. If we take two times of FT of signal f (t) it becomes to f (-t).

$$(Ƒ^2 f)(t) = (Ƒ(Ƒf))(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(w) e^{-i\omega t}\, dw = f(-t)$$

$Ƒ^2$ is called parity operator. Thus time axis rotated with an angleπ. Here F(w) is the FT of the f (t) signal. If we take three times of FT of the f (t), it is equivalent to F (-w). Thus

 ISBN: 978-81-932074-8-2

$$(F^3 f)(w) = (F(F^2 f))(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(-t) e^{-i\omega t}\, dt = F(-w)$$

Or

$$(F^3 f)(w) = (F(f)(-w) = F(-w)$$

This is the rotation 3π/2 with respect to the representation axis. Now four times of FT of the f(t) is $F^4(t) = t \; or \, F^4 = I$

IV. DEFINITION AND PROPERTIES OF FRFT

The FRFT of a function x with an anle α is defined as the function $R^{\alpha} = X_{\alpha}$

$$X_{\alpha}(u) = \int_{-\infty}^{\infty} x(t) K_{\alpha}(t,u)\, dt \qquad (1)$$

Where

$$K_{\alpha}(t,u) = \begin{cases} \sqrt{\frac{1-jcot\alpha}{2\pi}}\, e^{j\frac{t^2+u^2}{2}cot\alpha - jutcsc\alpha}, if\ \alpha\ is\ not\ multiple\ of\ \pi \\ \delta(t-u)\ ,\ \ if\ \alpha\ is\ multiple\ of\ 2\pi \\ \delta(t+u), if\ \alpha+\pi\ is\ multiple\ of\ \pi \end{cases} \qquad (2)$$

Basic properties of the FRFT:

Zero rotation- $R^0 = \mathrm{I}$ Identity operator
FT operator- $R^{\pi/2}$
Time reverse property- R^{π}
Inverse FT operator- $R^{3\pi/2}$
2π Rotation- $R^{2\pi}$=I
Additivity property- $R^{\beta}R^{\alpha} = R^{\alpha+\beta}$

Other properties-

$x(t-\tau)$	$\exp(j(\tau^2/2)\sin\alpha.\cos\alpha - ju\tau.\sin\alpha)\, X\alpha(u-\tau\cos\alpha)$
$x(t)\exp\,(jvt)$	$\exp\,(-jv^2(\sin\alpha\cos\alpha)/2 + jv\cos\alpha) X\alpha(u - v\sin\alpha)$
$x(t).t$	$u\cos\alpha\ X\alpha(u) + j\sin\alpha X'\alpha(u)$
$x(t)/t$	$-j\sec\alpha\ \exp(j(u^2/2)\cot\alpha) \int_{-\infty}^{u} x(z)\exp\,(-j(z^2/2)\cot\alpha) dz$

ISBN: 978-81-932074-8-2

$x(c.t)$	$\sqrt{\frac{1-jcot\ \alpha}{c^2-jcot\ \alpha}}\exp\ (j\ (u^2/2)\cot\alpha\ )(1-(cos^2\beta/cos^2\ \alpha))X\beta(\frac{u\ sin\ \beta}{c\sin\alpha})$ Where $\cot\beta=\frac{\cot\alpha}{c^2}$
$x'(t)$	$X'\alpha(u)\cos\alpha+ju\sin\alpha\ X\alpha(u)$
$\int_b^t x(t')dt'$	$\sec\alpha\ \exp\ (-j(u^2/2)\tan\ \alpha)\int_b^u X\alpha(z)\exp\ (j(z^2/2)\tan\ \alpha)dz$ If $\alpha-\pi/2\ is\ not\ a\ multiple\ of\ \pi.$

FRFT of Some Signals

SIGNAL	FRFT with order α
$\partial(t-\tau)$	$\sqrt{\frac{1-jcot\alpha}{2}}e^{j\frac{\tau^2+u^2}{2}cot\alpha-jcosec(\alpha u\tau)}$
$e^{-j(at^2+bt+c)}$	$\sqrt{\frac{1-jcot\alpha}{j2a-jcot\alpha}}e^{\frac{j(2acot\alpha-1)u^2}{2(cot\alpha-2a)}}.e^{\frac{j(2acot\alpha-1)u^2}{2(cot\alpha-2a)}}.e^{\frac{-jb^2}{2cot\alpha-2a}-jc}$
1	$\sqrt{1+jtan\alpha}.e^{\frac{-(ju^2tan\alpha)}{2}}$
$\cos(vt)$	$\sqrt{1+jtan\alpha}.e^{\frac{-(ju^2+v^2tan\alpha)}{2}}.\cos\ (uvsec\alpha)$
$\sin(vt)$	$\sqrt{1+jtan\alpha}.e^{\frac{-(ju^2+v^2tan\alpha)}{2}}.\sin(uvsec\alpha)$

Definition of Basic Properties-

1. **Zero rotation:** Zero rotation means when system is rotated by the angle α then there will no variation and the FRFT which is a function of x defined

   Equation (1), then the operator R^{α} value equal to the Identity Operator shown as:

   $R^{\alpha}=X_{\alpha}$

$$X_{\alpha}(u) = \int_{-\infty}^{\infty} x(t) K_{\alpha}(t,u)\, dt$$

$$X_0(u) = I$$

$$\therefore R^{\alpha} = I$$

2. **FT operator:** Fourier Transform Operator which can be realized by putting the value of $\propto = \frac{\pi}{2}$ in the operator $R^{\propto}$ then FRFT will become FT described as

$$X_{\pi/2}(u) = \int_{-\infty}^{\infty} x(t) K_{\pi/2}(t,u)\, dt$$

$$K_{\alpha}(t,u) = \sqrt{\frac{1 - j\cot\alpha}{2\pi}}\, e^{j\frac{t^2+u^2}{2}\cot\alpha - jut\csc\alpha}$$

$$K_{\pi/2}(t,u) = \frac{1}{\sqrt{2\pi}} e^{-jut}$$

$$K_{\pi/2}(t,u) = \int_{-\infty}^{\infty} 1/\sqrt{2\pi}\, e^{-jut}\, x(t)\, dt$$

$$\therefore K_{\frac{\pi}{2}}(t,u) = FT$$

3. **Additivity Property:** When two FR signals can be multiplied then the final result is equipped with operation of addition ofthe powers of variables. The equation can be described as-
If there are two operators R^{β} *and* R^{α} multiplication of both will give

$$R^{\beta} R^{\alpha} = R^{\alpha+\beta}$$

4. **2π Rotation:** One complete rotation of the signal corresponds to time- frequency plane will give out identity operator similar to that of zero rotation because after $360°$ rotation the values are repeated. Therefore

$$R^{2\pi} = \mathrm{I}$$

So, by taking any figure as a reference for the FRFT signal different graphs can be obtained by changing the value of α which is obtained from the Matlab by programming the code for it.

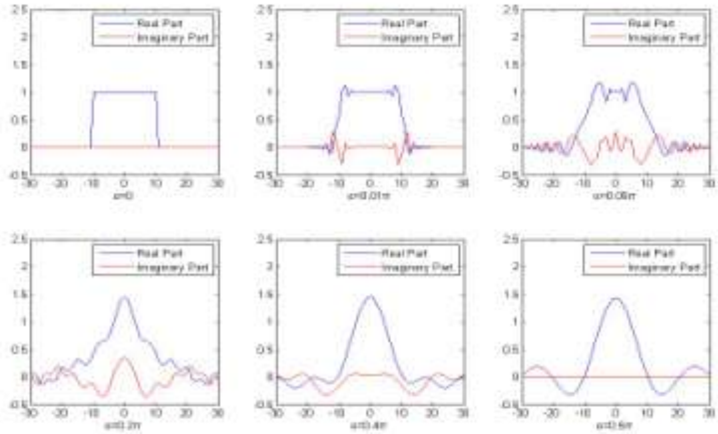

Figure2:-FRFT of Rectangle with different values of α

From the above figures we conclude that we will get direct result if we apply transform method in order to convert a rectangular signal into a sinc signal. By using FRFT we will find out that there are few steps in between them instead of getting directly a sinc signal by using different values of α as there exist a relation between a and α. It became easy to obtain results for different orders. This is basically used for signal security and we can find better result for noise removal using FRFT. The method of rotation in FTFT is used for the removal of noise by using above concept, there is a pictorial representation which shows how noise will be removed using FRFT-

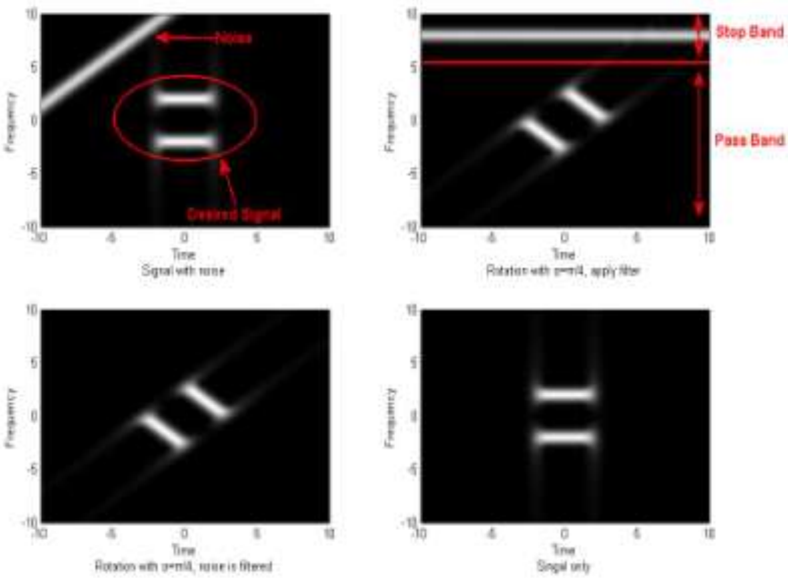

Figure3:- Noise removal from signal using FRFT

V. APPLICATION OF THE FRFT

There are various applications of the Fractional Fourier Transform such as-

1. Image Compression

2. OFDM
3. Target tracking
4. Signal processing and Image processing
5. Digital Watermarking
6. Image Encryption with Multiorders FRFT

1. **Image Compression:-**Image compression is the application of data compression on digital images. In effect, the objective is to reduce redundancy of the image data in order to be able to store or transmit data in an efficient form.

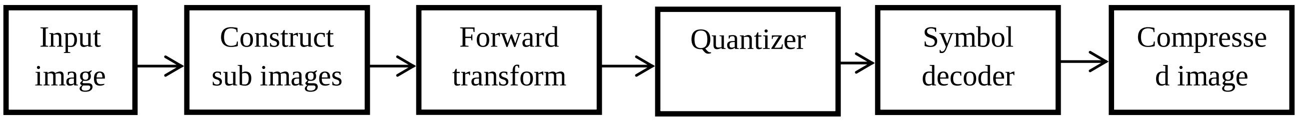

Figure4–Encoding of image

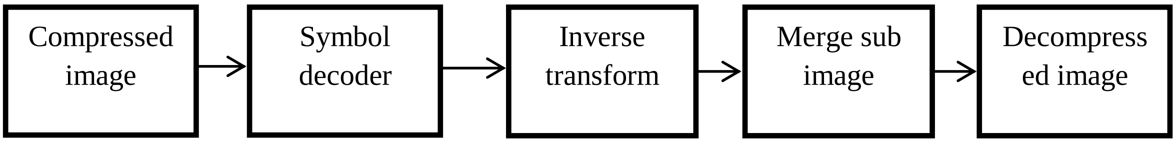

Figure5- Decoding of image

The entire compression process is divided into two sections:

i. Encoding Of Image
ii. Decoding Of Image

The input image is first converted into small sub image or cells and then transform method is applied to every sub images-this approach also lead to the security of theimage and compression to a great extent. Quantizer is used for the process of quantization of bits and after decoding it we will obtain a compressed image. The process of deciding is just the reverse of encoding , in this the obtained compressed image is applied to the symbol decoder and using inverse FRFT we will obtain an decompressed image after combining all the sub image into one image

Image compression [4] can be lossy or lossless. Lossless compression is sometimes preferred for medical imaging, technical drawing, icons or comics. This is because lossy compression method especially when used at low bit rates, introduce compression artifacts. Lossless compression method may also be preferred for high value content, such as medical imaginary or image scans made for archival purpose. Lossy methods are especially suitable for natural image such as photos in application where minor loss of fidelity is acceptable to achieve a substantial reduction in bit rate. The lossy compression that produces imperceptible difference can be called visually lossless

Some Image compression characteristics:

- Compression ratio(CR)
- Compression speed(CS)

- Image quality

Image quality basically depends upon two parameters namely:

a) Mean square error (MSE)

b) Peak Signal To Noise Ratio (PSNR)

These can be defined as-

$$\mathrm{MSE} = \frac{1}{\mathrm{MN}} * \sum_{i=0}^{M_1} \sum_{i=0}^{N-1} [f'(i,j) - f(i,j)]^2 \quad (3)$$

$$\mathrm{PSNR} = 10\log_{10}\left[\frac{\mathrm{M*N}}{\mathrm{MSE}}\right] \mathrm{dB} \quad (4)$$

Where $f'(i,j)$the function of is decoded image and $f(i,j)$ is function of original image.

2. OFDM (Orthogonal Frequency Division Multiplexing):-

It is a method ofencoding digital data on multiple carrier frequencies. Orthogonal frequency division multiplexing (OFDM) has recently become a key modulation technique for both broadband wireless and wire-line applications. It has been adopted for digital audio broadcasting (DAB) and digital terrestrial television broadcasting (DVB). OFDM is a special case of Multicarrier transmission, where a single data stream is transmitted over number of lower rate Subcarrier. The problem of intersymbol interference (ISI) introduced by multipath channel is significantly reduced in OFDM by using the cyclic prefix (CP) as a guard interval between OFDM blocks. To reduce the effect of doppler frequency spread, the FRFT Multi carrier system is introduced. As we know that transmitting signal consist of many blocks and each start with a cyclic prefix (CP) to reduce interblock interferrence.By using FRFT SIR i.e. Signal to Noise Interferrence Ratio is superior or accurate by choosing optimal fractional factor this is possible only when frequency offest exists. In today's wireless communication by mobile phones, the channel frequency response can be rapidly time varying, in which case the Doppler spread may not be neglected and cause inter-channel interference. Choosing chirp carriers instead of time or frequency modulation can overcome this problem. Performance of this technique is significantly improved since the time-frequency plane can be adjusted (rotated) in a way to compensate undesired modulation of the signals introduced.

3. Target Tracking:-Target Tracking or detection basically means finding the correct location of a stationary or moving object with the help of special type of devices. Target Tracking [9] is commonly used in military, marine's, medical imaging etc. Detection of moving target in a sea plays an important role in military & marine fields. The echo of weak moving target in a sea consist of a strong sea clutter which will cause a serious false alarm due to many reflecting angles from sea waves and led the detection of an object a difficult task.

The above problem will be resolved by combining two methods Wavelet Based Approach(WBA) which have a great potential in signal detection and an important algorithm i.e. FRFT approach. A grading iterative computation method is carried out to get better accuracy and faster calculation. Grading iterative computation method is used to search for the strongest peak value. Then, the signal is restructured with the estimated parameters. By the "CLEAN" technique, parameters of multi-component LFM signals can be detected and estimated in turn.At the end simulation is done to verify the real parameters with the observed parameters. The target tracking model for detection-

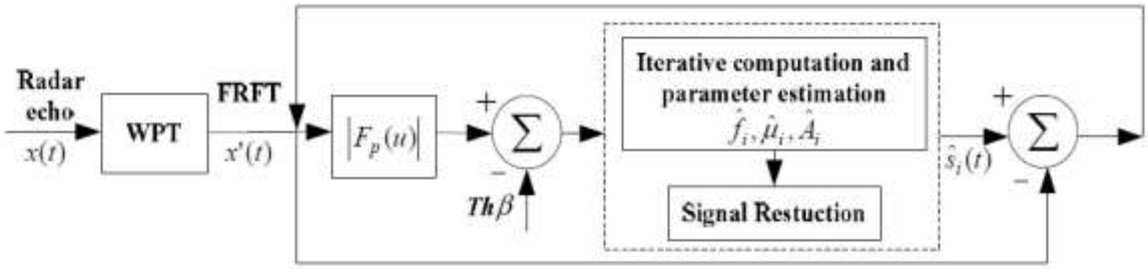

Figure6- Target Tracking Detection Model

4. Signal Processing & Image Processing:-Signal processing is a method that encompasses the fundamental theory, applications, algorithms and implementations of processing or transferring information in different forms. It uses mathematical, statistical, computational techniques for representation modeling, analysis, security etc. FRFT is used in image processing as image encryption and decryption. FRFT algorithm is applied to the input image which is almost similar to the process of image compression. In signal-processing application, it is basically used for filtering, signal recovery, signal reconstruction, signal synthesis, beamforming, signal detectors, correlators, image recovery, restoration and enhancement, pattern recognition.

5. Filtering using FRFT

Filtering is a method of removal of noise from the desired signal to enhance the signal strength, so that it can recover easily at the receiver end. This is discussed in the figure 3 how noise can be removed using FRFT. The received signal can be observed with some finite accuracy determined by noise or other errors. Signal restoration problems are signal recovery problems where the received signal is a distorted, noisy or otherwise degraded version of the transmitted signal. In signal synthesis a desired output signal is specified and input to the system is to be chosen so that the required signal is observed at the output. All these inverse problems are mathematically similar. In each case the problem is to estimate the input from knowledge of output, also using any available prior knowledge regarding the nature of the input and/or the nature and statistics of the measurement error or noise or the specified tolerance [14][17]. The concept of filtering in fractional Fourier domain is to realize, flexibly and efficiently, general shift-variant linear filters for a variety of applications. Figure 1 shows Wigner distribution of a desired signal and noise superimposed in single plot. It is clear from here that signals with significant overlap in both time and frequency domains may have little or no overlap in a fractional Fourier domain. A frequently used criterion for optimal filtering is mean square error (MSE).The optimal filtering can be obtained depending upon the criteria of optimization. The main criteria of optimization are minimum mean squared error (MMSE), maximum signal-to-noise ratio (SNR) and minimum variance. Each criterion has its own advantages and disadvantages. The objective is to recover the desired signal free from noise and fading in the received signal, in stationary and moving source problems. Let the filter input be $y(t)$ and the reference signal be $x(t)$. The weights of the filter can be chosen in order to minimize the MSE between the output and the reference signal.

$$J(w) = E\{\| y(t) - x(t) \|^2\}$$

where $\|\cdot\|$ is the $L2$ norm given by $\| y(t) \|^2 = \int_{-\infty}^{\infty} y(t) y*(t) dt$.The optimum weights can befound by setting the derivative of J(w) to w* equal to zero. They are given as

$$w_{opt} = R_y^{-1} r_{yx},$$

where$R_y$is the covariance matrix of the received signal and $r_{yx}$is the cross-covariance between the input of the filter and the desired signal. Figure 2 shows that the MSE is less in the case of $a = -0.3$ domain (optimum FRFT domain) as compared to $a = 0$ (time domain) and $a = 1$(frequency domain). So filtering is to be done in optimum FRFT domain for least MSE.

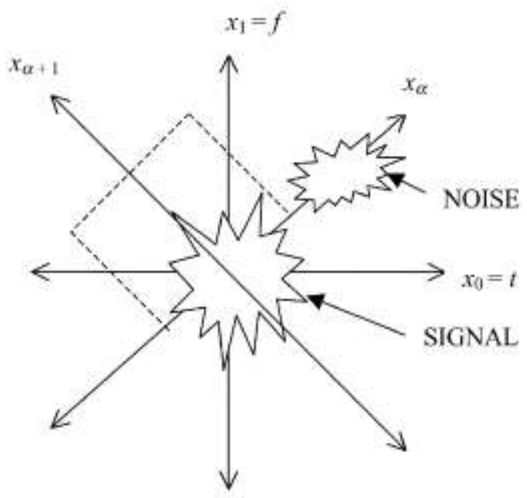

Figure 7- Filtering in fractional Fourier Domain in time-frequency plane

5. Digital watermarking:-Digital Watermarking is basically a phenomenon by which we can easily encrypt and decrypt a data in digital format so that it can be used by authorized users and unauthorized users will not be able to decrypt the data.To authenticate an image, a watermark is embedded in the signal. Embedding a hidden stream of bits in a file is called Digital Watermarking. The file could be an image, audio, video or text. Nowadays, digital watermarking has many applications such as broadcast monitoring, owner identification, proof of ownership, transaction tracking and content authentication. It should be difficult to detect for an outsider and it should not disturb the image visually. In [17] the following procedure is proposed. First perform a 2-dimensional FRFT on the image. Then sort the resulting coefficient's and add some watermark key to a subsequence of the coefficients. That should be not the largest coefficient's, in order not to disturb the image, but also not to the smallest ones, because these could be drowned in noise and filtered out by a noise filter.There are basically two techniques or models that we have developed for the encryption and decryption of digital Watermark.

These are:

- Watermarking without side-information.

- Watermarking with side-information.

The digital watermark can be obtained from Matlab which is shown below-

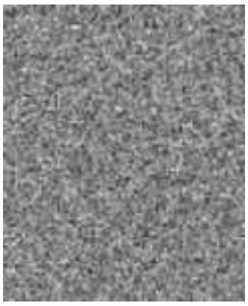

Figure 8- Showing Digital Watermarking

6. Image Encryption with Multiorder FRFT:-The traditional information security systems based on the FRFT are only the rotation of the corresponding Fourier based systems in the time-frequency plane, i.e., the FT or the Fourier domain (FD) is replaced by the FRFT or the fractional Fourier domain (FRFD). Basically FRFT can be utilised for security of the information which is done in encrypted and decrypted form. The original information in the existing security system based on the fractional Fourier transform (FRFT) is essentially protected by only a certain order of FRFT.

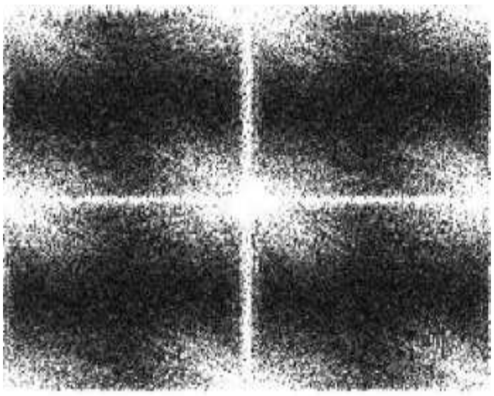

Figure9-Encrypted pattern

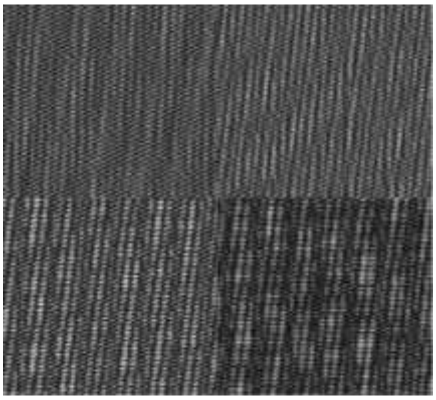

Figure10- Decrypted pattern

In this paper we basically discuss about generalisation of FRFT and its application in different sector including the importance of security of information contents. For this, first of all an image is divide into equal cells and then encrypted image is obtained by the summation of different ordersof inverse discrete fractional Fourier transform (IDFRFT) of the interpolated sub images.

Thus, the original image is protected by multiorders of FRFT. The transform orders as the security keys, the proposed method is with a larger key space than the existing image encryption systems based on the FRFT. The encrypted image is obtainedby the summation of different orders of IDFRFT of the interpolated sub images. The whole transform orders of the utilized FRFT are used as the secret keys for the decryption of each sub image. It is verified by the experimental results that the image decryption is highly sensitive to the deviations in the transform orders. Compared with the traditional image encryption methods based on the FRFT, the proposed method is with a larger key space and the amount of keys can be set as large as two times the amount of the pixels in the original image.

REFERENCES

[1] R.N.Bracewell, The Fourier transforms and its applications, McGraw-Hill (1986).

[2] Wiener N. Hermitian polynomials and Fourier analysis. J Math Phys MIT, 1929, 18: 70－73

[3] V. Namias, “The fractional order Fourier transform and its application to quantummechanics,” *J. Inst. Maths. Applics.*, vol. 25, pp. 241-265, 1980.

[4] L. B. Almeida, “The fractional Fourier transform and time-frequencyrepresentations,” *IEEE Trans. Signal Processing*, vol. 42, no. 11, pp. 3084-3091,Nov. 1994.

[5] H.M. Ozaktas, Z. Zalevsky, and M.A. Kutay. The fractional Fourier transform. Wiley,Chichester, 2001.

[6] S.C. Pei and J.J. Ding. Two-dimensional plane generalized fractional Fourier transform.IEEE Trans. Sig. Proc., 49(4):878{897, 2001}.

[7] S.C. Pei and J.J. Ding, Relations between fractional operations and time-frequencydistributions and their applications. IEEE Trans. Sig. Proc., 49(8):1638{1655, 2001.

[8] E.U. Condon. Immersion of the Fourier transform in a continuous group of functionaltransformations, Proc. National Academy Sciences, 23:158{164, 1937}.

[9] A Fast FRFT Based Detection Algorithm of Multiple Moving Targets in Sea Clutter-Xiaolong Chen and Jian Guan Department of Electronic and Information Engineering Navy Aeronautical and Astronautical University, NAAU Yantai, Shandong P.R. China, 264001

[10] A.C. McBride and F.H. Kerr. On Namias's fractional Fourier transforms. IMA J. Appl. Math., 39:159{175, 1987.

[11] I.S. Yetik, M.A. Kutay and H.M. Ozaktas, “ImageRepresentation and Compression Using Fractional FourierTransform”, *Optics Communicatio,n*vol. 197, pp 275-278,2001.

[12] R. Saxena and K. Singh, “Fractional Fourier Transforms: A Review,” IETE Journal of education, vol. 48, no. 1, pp 13-29,2007.

[13] Rajiv Saxena ,Kulbir Singh. “Fractional Fourier transform: A noveltool for signal processing” *J. Indian Inst. Sci.*, Jan.–Feb. 2005, 85,11–26.

[14] H. M. Ozaktas, O. Arikan, M. A. Kutay and G. Bozdagi, Digital computation of the fractional Fourier transforms, *IEEE Trans. Signal Processing*, 44, 2141–2150 (1996).

[15] I. Djurovi_c, S. Stankovi_c, and I. Pitas. Digital watermarking in the fractional Fouriertransform domain. J. of Network and Computer Applications, 24:167{173, 2001.

[16] M. F. Edren, M. A. Kutay and H. M. Ozaktas, Repeated filtering in consecutive fractional Fourier domains and its applications to signal restoration, *IEEE Trans. Signal Processing*, 47, 1458–1462 (1999).