CIRIC TYPECOINCIDENCE & FIXED POINT RESULTS FOR NON EXPANSIVE SINGLE VALUED MAPS IN 2-METRIC SPACES

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ABSTRACT

In this paper, we consider the existence of coincidences and fixed points of non-expansive type conditions satisfied by single valued maps and prove some fixed point theorems for non-expansive type single valued mappings in 2-metric space.

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I INTRODUCTION

Fixed point theorems for contractive, non-expansive, contractive type and non-expansive type mappings provide techniques for solving a variety of applied problems in mathematical and engineering sciences. It is one of the reason that many authors have studied various classes of contractive type or non-expansive type mappings. If T is such that for all x, y in X

(1.1) $$d(Tx, Ty) \leq \lambda d(x, y)$$

where $0 < \lambda < 1$, then T is said to be a contraction mapping. If T satisfies (1.1) with $\lambda = 1$, then T is called a non-expansive mapping. If T satisfies any conditions of type

(1.2) $$d(Tx, Ty) \leq a_1 d(x, y) + a_2 d(x, Tx) + a_3 d(y, Ty) + a_4 d(x, Ty) + a_5 d(y, Tx)$$

where a_i $(i = 1,2,3,4,5)$ are nonnegative real numbers such that $a_1 + a_2 + a_3 + a_4 + a_5 < 1$, then T is said to be a contractive type mapping. If T satisfies (1.2) with $a_1 + a_2 + a_3 + a_4 + a_5 = 1$, then T is said to be a non-expansive type mapping. Similar terminology is used for multi-valued mappings.

Bogin [3] proved the following result:

Theorem 1.1 Let X be a nonempty complete metric space and $T: X \to X$ a mapping satisfying

(1.3) $$d(Tx, Ty) \leq ad(x, y) + b[d(x, Tx) + d(y, Ty)] + c[d(x, Ty) + d(y, Tx)]$$

where $a \geq 0, b > 0, c > 0$ and

(1.4) $$a + 2b + 2c = 1$$

Then T has a unique fixed point.

This result was generalized by Rhoades [19] and Ciric [6,7]. Iseki [13] studied a family of commuting mappings $T_1, T_2, \dots T_n$ which satisfy (1.3) with $a \geq 0, b \geq 0, c \geq 0$ and $a + 2b + 2c = 1$. For Banach spaces the famous is Gregus's Fixed Point Theorem [11] for non-expansive type single-valued mappings, which satisfy (1.3) with $c = 0, a < 1$. Ciric [6] introduced and investigated a new class of self-mappings T on X which satisfy an inequality of type (1.3) with $b \geq 0$ and still have a fixed point. Also proved that by an example if the mapping T satisfies (1.3) with $b = 0$ and if a and c are such that (1.4) holds, then T need not have a fixed point. Therefore, a contractive condition for T , which shall guarantee a fixed point of T in the case $b = 0$ and $a + 2c = 1$, must be stricter then (1.3).

The concept of 2-metric space is a natural generalization of the classical one of metric space. It has been investigated, initially, by Gahler and has been developed extensively by Gahler and many other mathematicians [8-10]. The topology induced by 2-metric space is called 2-metric topology, which is generated by the set of all open spheres with two centers. Iseki [12] studied the fixed point theorems in 2-metric spaces. A number of fixed point theorems has been proved for 2-metric spaces. Liu and Zhang [15] proved a few necessary and sufficient conditions for the existence of a common fixed point of a pair of mappings in 2-metric spaces. These results have generalized and improved by a number of mathematicians. Singh, Adiga and Giniswami [20] proved a fixed point theorem in 2-metric spaces for non-expansive type mappings.

We recall the following definitions and results which can be found in [8].

Definition 2.1 (see [8]) Let X be a nonempty set. A real valued function d on X^3 is said to a 2-metric if, for all $x, y, z, u \in X$, the following conditions hold:

(1) To each pair of distinct points x, y in X, there exists a point $z \in X$ such that $d(x, y, z) \neq 0$;

(2) $d(x, y, z) = 0$ if at least two of x, y, z are equal;

(3) $d(x, y, z) = d(x, z, y) = d(y, x, z) = d(y, z, x) = d(z, x, y) = d(z, y, x)$;

(4) $d(x, y, z) \le d(x, y, u) + d(x, u, z) + d(u, y, z)$.

Then (X, d) is called a 2-metric space which will be sometimes denoted by X if there is no confusion. Every member $x \in X$ is called a point in X. Geometrically a 2-metric $d(x, y, z)$ represents the area of a triangle with vertices x, y and z.

Definition 2.2 A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be convergent to a point $x \in X$, if $lim_{n\to\infty} d(x_n, x, u) = 0$ for all $u \in X$.

Definition 2.3 A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be Cauchy sequence if for all $z \in X$, $lim_{n\to\infty} d(x_n, x_m, z) = 0$.

Definition 2.4 A sequence $\{x_n\}$ in a 2-metric space (X, d) is said to be complete if every Cauchy sequence in X is convergent.

Definition 2.5 (see [16]) Let f and T be mappings from a 2-metric space (X, d) into itself. The pair (f, T) is said to be compatible pair (co. p.) if for all u ∈ X, $lim_{n\to\infty} d(fTx_n, Tfx_n, u) = 0$ whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} fx_n = lim_{n\to\infty} Tx_n = p$ for some $p \in X$.

Definition 2.6 (see [16]) Let f and T be mappings from a 2-metric space (X, d) into itself. The pair (f, T) is said to be weakly compatible pair (w. co. p.) if $fx = Tx$ (for some $x \in X$) implies $fTx = Tfx$.

Definition 2.7 (see [16]) Let f and T be mappings from a 2-metric space (X, d) into itself. The pair (f, T) is said to be compatible of type (A) if

$$lim_{n\to\infty} d(Tfx_n, ffx_n, u) = lim_{n\to\infty} d(fTx_n, TTx_n, u) = 0$$

for all $u \in X$, whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} Sx_n = lim_{n\to\infty} Tx_n = p$ for some $p \in X$.

Definition 2.8 (see [16]) Let f and T be mappings from a 2-metric space (X, d) into itself. The pair (f, T) is said to be weakly compatible of type (A) if

$$lim_{n\to\infty} d(fTx_n, TTx_n, z) \le lim_{n\to\infty} d(Tfx_n, TTx_n, z)$$

and

$$lim_{n\to\infty} d(Tfx_n, ffx_n, z) \le lim_{n\to\infty} d(fTx_n, ffx_n, z)$$

for all $z \in X$, whenever $\{x_n\}$ is a sequence in X such that $lim_{n\to\infty} Sx_n = lim_{n\to\infty} Tx_n = t$ for some $t \in X$.

Example: 2.9 (see [17]) Define d on $[0,1] \times [0,1] \times [0,1]$ by $d(x,y,z) = min\{\rho(x,y), \rho(y,z), \rho(z,x)\}$, where ρ is a usual metric on $[0,1]$. Then it is easy to see that d is a 2-metric on $[0,1]$. Define $f, T: [0,1] \to [0,1]$ by $fx = \frac{x}{1+x}$ and $Tx = \frac{x}{2}$. Choose a sequence $\{x_n\}$ in $[0,1]$ such that converges to zero in $[0,1]$ i.e. $x_n \to 0 \in [0,1]$ as $n \to \infty$. Then for all $u \in X$,

$$lim_{n\to\infty} d(fTx_n, Tfx_n, u) = lim_{n\to\infty} d\left(f\left(\frac{x_n}{2}\right), T\left(\frac{x_n}{1+x_n}\right), u\right)$$

$$= lim_{n\to\infty} d\left(\frac{\left(\frac{x_n}{2}\right)}{1+\left(\frac{x_n}{2}\right)}, \frac{\left(\frac{x_n}{1+x_n}\right)}{2}, u\right)$$

$$= lim_{n\to\infty} d\left(\frac{x_n}{2+x_n}, \frac{x_n}{2(1+x_n)}, u\right)$$

$$= lim_{n\to\infty} min\left\{\rho\left(\frac{x_n}{2+x_n}, \frac{x_n}{2(1+x_n)}\right), \rho\left(\frac{x_n}{2(1+x_n)}, u\right), \rho\left(u, \frac{x_n}{2+x_n}\right)\right\}$$

$$\leq lim_{n\to\infty} \rho\left(\frac{x_n}{2+x_n}, \frac{x_n}{2(1+x_n)}\right)$$

$$= lim_{n\to\infty} x_n \left|\frac{1}{2+x_n} - \frac{1}{2(1+x_n)}\right| = 0.$$

Thus, $d(fTx_n, Tfx_n, u) \to 0$, as $n \to +\infty$ when $x_n \to 0 \in X$ as $n \to +\infty$. Hence (f,T) is a co. p. In view of Proposition 2.4 of [18], every pair of compatible mappings of type (A) is weakly compatible mappings of type (A) whereas in view of Proposition 2.9 of [18], every pair of compatible mappings of type (A) is weakly compatible pair.

In this paper, we prove Ciric [6] type common fixed point theorems under non-expansive type conditions in the setting of 2-metric spaces. We shall investigate a class of self-mappings T, f on X which satisfy the following non-expansive type condition:

(1.5) $$d(Tx,Ty,u) \leq a(x,y)\ max\{d(fx,fy,u), d(fx,Tx,u), d(fy,Ty,u), \tfrac{1}{2}[M(x,y,u) + m(x,y,u)]\}$$

$$+c(x,y)[M(x,y,u) + hm(x,y,u)]$$

for all $x, y, u \in X$, where

$M(x,y,u) = max\{d(fx,Ty,u), d(fy,Tx,u)\}$

$m(x,y,u) = min\{d(fx,Ty,u), d(fy,Tx,u)\}$

and

(1.6) $$0 < h < 1, a(x,y) > 0,\ \beta = inf\{c(x,y) : x,y \in X\} > 0$$

(1.7) $$sup_{x,y\in X}\big(a(x,y) + 2c(x,y)\big) = 1.$$

II MAIN RESULT

Now, we give our main results.

Theorem 2.1 Let (X,d) be a 2-metric space, T,f are self maps of X satisfying condition (1.5), where a and c satisfying (1.6) and (1.7) with $T(X) \subseteq f(X)$ and either (a) X is complete and f is surjective; or (b) X is complete, f is continuous and T,f are compatible; or (c) $f(X)$ is complete; or (d) $T(X)$ is complete. Then f and T have a coincidence point in X. Further, the coincidence value is unique, i.e. $fp = fq$ whenever $fp = Tp$ and $fq = Tq, (p,q \in X)$.

Proof Let $x_0 \in X$. We construct two sequences $\{x_n\}$ and $\{y_n\}$ as follows: Since $T(X) \subseteq f(X)$, choose x_1 so that $y_1 = fx_1 = Tx_0$. In general, choose x_{n+1} so that $y_{n+1} = fx_{n+1} = Tx_n,\ n \in \mathbb{N}$. For simplicity, we set $d_n(u) = d(y_n, y_{n+1}, u)$ for all $u \in X$ and $n \in \mathbb{N} \cup \{0\}$. Obviously, $d_n(y_n) = 0 = d_n(y_{n+1})\ \forall\ n \in \mathbb{N} \cup \{0\}$.

First, we claim that $d_n(y_{n+2}) = 0$. On the contrary, suppose that $d_n(y_{n+2}) \neq 0, \forall\ n \in \mathbb{N} \cup \{0\}$. Applying (1.5), we have

(2.1) $$d_n(y_{n+2}) = d(y_n, y_{n+1}, y_{n+2}) = d(Tx_n, Tx_{n+1}, y_n)$$

$$\leq a\, max\{d(fx_n, fx_{n+1}, y_n), d(fx_n, Tx_n, y_n), d(fx_{n+1}, Tx_{n+1}, y_n)$$

$$, \tfrac{1}{2}[M(x_n, x_{n+1}, y_n) + m(x_n, x_{n+1}, y_n)]\} + c[M(x_n, x_{n+1}, y_n) + h\, m(x_n, x_{n+1}, y_n)]$$

$$= a\, max\{d(y_n, y_{n+1}, y_n), d(y_n, y_{n+1}, y_n), d(y_{n+1}, y_{n+2}, y_n)$$

$$, \tfrac{1}{2}[M(x_n, x_{n+1}, y_n) + m(x_n, x_{n+1}, y_n)]\} + c[M(x_n, x_{n+1}, y_n) + h\, m(x_n, x_{n+1}, y_n)]$$

where a and c are evaluated at (x_n, x_{n+1}).

Since

$$m(x_n, x_{n+1}, y_n) = min\{d(fx_n, Tx_{n+1}, y_n), d(fx_{n+1}, Tx_n, y_n)\}$$

$$= min\{d(y_n, y_{n+2}, y_n), d(y_{n+1}, y_{n+1}, y_n)\} = 0$$

And

$$M(x_n, x_{n+1}, y_n) = max\{d(fx_n, Tx_{n+1}, y_n), d(fx_{n+1}, Tx_n, y_n)\}$$

$$= max\{d(y_n, y_{n+2}, y_n), d(y_{n+1}, y_{n+1}, y_n)\} = 0.$$

Hence from (2.1), we have

$d_n(y_{n+2}) \leq a d_n(y_{n+2})$

From (1.6), we have

(2.2) $$d_n(y_{n+2}) = 0.$$

We shall prove that $\{d_n(u)\}_{n\in\mathbb{N}\cup\{0\}}$ is a non-increasing sequence in $\mathbb{R}^+$. For all $u \in X$, on the contrary, assume that $d_{n+1}(u) > d_n(u)$. Again applying (1.5), we have

(2.3) $$d_{n+1}(u) = d(y_{n+1}, y_{n+2}, u) = d(Tx_n, Tx_{n+1}, u)$$

$$\leq a\, max\{d(fx_n, fx_{n+1}, u), d(fx_n, Tx_n, u), d(fx_{n+1}, Tx_{n+1}, u)$$

$$, \tfrac{1}{2}[M(x_n, x_{n+1}, u) + m(x_n, x_{n+1}, u)]\} + c[M(x_n, x_{n+1}, u) + h\, m(x_n, x_{n+1}, u)]$$

$$= a\, max\left\{d_n(u), d_n(u), d_{n+1}(u), \tfrac{1}{2}[M(x_n, x_{n+1}, u) + m(x_n, x_{n+1}, u)]\right\}$$

$$+ c[M(x_n, x_{n+1}, u) + h\, m(x_n, x_{n+1}, u)]$$

where a and c are evaluated at (x_n, x_{n+1}).

Since

$$m(x_n, x_{n+1}, u) = min\{d(fx_n, Tx_{n+1}, u), d(fx_{n+1}, Tx_n, u)\}$$

$$= min\{d(y_n, y_{n+2}, u), d(y_{n+1}, y_{n+1}, u)\} = 0$$

$$M(x_n, x_{n+1}, u) = max\{d(fx_n, Tx_{n+1}, u), d(fx_{n+1}, Tx_n, u)\}$$

$$= max\{d(y_n, y_{n+2}, u), d(y_{n+1}, y_{n+1}, u)\}$$

$$= d(y_n, y_{n+2}, u).$$

Then from (2.2), we have

$$M(x_{n+1}, x_{n+2}, u) = d(y_n, y_{n+2}, u)$$

$$\leq d(y_n, y_{n+2}, y_{n+1}) + d(y_n, y_{n+1}, u) + d(y_{n+1}, y_{n+2}, u)$$

$$\leq 2d_{n+1}(u)$$

The inequality (2.3) gives

$$d_{n+1}(u) \leq (a+2c)\, d_{n+1}(u)$$

a contradiction. Thus, our supposition that $d_{n+1}(u) > d_n(u)$ was wrong. Therefore $\{d_n(u)\}_{n\in\mathbb{N}\cup\{0\}}$ is a non-increasing sequence of non-negative real numbers. Therefore, for all n we have

(2.4) $$d_{n+1}(u) \leq d_n(u).$$

Now, we shall prove that for any $n, m \in \mathbb{N}$, $d_n(y_m) = 0$.

Let $n, m \in \mathbb{N}$ and if $n \geq m$ and $u = y_m$, then from (2.4), we have

(2.5) $$d_n(y_m) \leq d_{n-1}(y_m) \leq \cdots \leq d_m(y_m) = 0.$$

If $n < m$, then from (2.5), we have

$$d_n(y_m) = d(y_n, y_{n+1}, y_m)$$

$$\leq d(y_n, y_{n+1}, y_{m-1}) + d(y_n, y_{m-1}, y_m) + d(y_{m-1}, y_{n+1}, y_m)$$

$$= d_n(y_{m-1}) + d_{m-1}(y_n) + d_{m-1}(y_{n+1})$$

$$= d_n(y_{m-1}) \leq d_n(y_{m-2}) \leq \cdots \leq d_n(y_{n+1}) = 0.$$

Thus for any $n, m \in \mathbb{N}$,

(2.6) $$d_n(y_m) = 0$$

Next we shall prove that $d(y_i, y_j, y_k) = 0$ for all $i, j, k \in \mathbb{N}$. Without loss of generality, we may assume that $i \leq j$, it follows that

$$d(y_i, y_j, y_k) \leq d(y_i, y_j, y_{i+1}) + d(y_i, y_{i+1}, y_k) + d(y_{i+1}, y_j, y_k)$$

$$= d_i(y_j) + d_i(y_k) + d(y_{i+1}, y_j, y_k)$$

$$= d(y_{i+1}, y_j, y_k)$$

Similarly,

$$d(y_{i+1}, y_j, y_k) \leq d(y_{i+2}, y_j, y_k)$$

Inductively, we have

(2.7) $$d(y_i, y_j, y_k) \leq d(y_{j-1}, y_j, y_k) = d_j(y_k) = 0$$

We claim that $\lim_{n\to\infty} d_n(u) = 0$.

Applying (1.5), we have

(2.8) $$d(y_{n-1}, Tx_n, u) = d(Tx_{n-2}, Tx_n, u)$$

$$\leq a\, max\{d(fx_{n-2}, fx_n, u), d(fx_{n-2}, Tx_{n-2}, u), d(fx_n, Tx_n, u)$$

$$, \tfrac{1}{2}[M(x_{n-2}, x_n, u) + m(x_{n-2}, x_n, u)]\} + c[M(x_{n-2}, x_n, u) + hm(x_{n-2}, x_n, u)]$$

$$= a\, max\{d(y_{n-2}, y_n, u), d(y_{n-2}, y_{n-1}, u), d(y_n, y_{n+1}, u)$$

$$, \tfrac{1}{2}[M(x_{n-2}, x_n, u) + m(x_{n-2}, x_n, u)]\} + c[M(x_{n-2}, x_n, u) + hm(x_{n-2}, x_n, u)]$$

where a and c are evaluated at (x_{n-2}, x_n).

Since by using (2.4), (2.5) and (2.6), we have

$$d(y_{n-2}, y_n, u) \leq d(y_{n-2}, y_n, y_{n-1}) + d(y_{n-2}, y_{n-1}, u) + d(y_{n-1}, y_n, u)$$

$$\leq d_{n-2}(y_n) + d_{n-2}(u) + d_{n-1}(u)$$

$$\leq 2d_{n-2}(u)$$

$$d(y_{n-2}, y_{n+1}, u) \leq d(y_{n-2}, y_{n+1}, y_{n-1}) + d(y_{n-2}, y_{n-1}, u) + d(y_{n-1}, y_{n+1}, u)$$

$$\leq d(y_{n-2}, y_{n+1}, y_{n-1}) + d_{n-2}(u) + d(y_{n-1}, y_{n+1}, y_n) + d(y_{n-1}, y_n, u)$$

$$+ d(y_n, y_{n+1}, u)$$

$$= d_{n-2}(u) + d_{n-1}(u) + d_n(u)$$

$$\leq 3d_{n-2}(u)$$

$$m(x_{n-2}, x_n, u) = min\{d(fx_{n-2}, Tx_n, u), d(fx_n, Tx_{n-2}, u)\}$$

$$= min\{d(y_{n-2}, y_{n+1}, u), d(y_n, y_{n-1}, u)\}$$

$$\leq min\{3d_{n-2}(u), d_{n-1}(u)\}$$

$$= d_{n-1}(u)$$

$$M(x_{n-2}, x_n, u) = max\{d(fx_{n-2}, Tx_n, u), d(fx_n, Tx_{n-2}, u)\}$$

$$= max\{d(y_{n-2}, y_{n+1}, u), d(y_n, y_{n-1}, u)\}$$

$$\leq max\{3d_{n-2}(u), d_{n-1}(u)\} = 3d_{n-2}(u)$$

Using above all inequalities and (1.7), the inequality (2.8) gives

(2.9) $$d(y_{n-1}, y_{n+1}, u) \leq a \max\left\{2d_{n-2}(u), d_{n-2}(u), d_n(u), \tfrac{1}{2}[3d_{n-2}(u) + d_{n-1}(u)]\right\}$$

$$+c[3d_{n-2}(u) + hd_{n-1}(u)]$$

$$\leq [2a + c(3 + h)]d_{n-2}(u)$$

$$= [2 - c(1 - h)]d_{n-2}(u)$$

Again from (1.5), we have

(2.10) $$d(y_n, y_{n+1}, u) = d(Tx_{n-1}, Tx_n, u)$$

$$\leq a\, max\{d(fx_{n-1}, fx_n, u), d(fx_{n-1}, Tx_{n-1}, u), d(fx_n, Tx_n, u)$$

$$, \tfrac{1}{2}[M(x_{n-1}, x_n, u) + m(x_{n-1}, x_n, u)]\big\} + c[M(x_{n-1}, x_n, u) + hm(x_{n-1}, x_n, u)]$$

$$= a\, max\{d(y_{n-1}, y_n, u), d(y_{n-1}, y_n, u), d(y_n, y_{n+1}, u)$$

$$, \tfrac{1}{2}[M(x_{n-1}, x_n, u) + m(x_{n-1}, x_n, u)]\big\} + c[M(x_{n-1}, x_n, u) + hm(x_{n-1}, x_n, u)]$$

where a and c are evaluated at (x_{n-1}, x_n).

Since

$$m(x_{n-1}, x_n, u) = min\{d(fx_{n-1}, Tx_n, u), d(fx_n, Tx_{n-1}, u)\}$$

$$= min\{d(y_{n-1}, y_{n+1}, u), d(y_n, y_n, u)\} = 0$$

$$M(x_{n-2}, x_n, u) = max\{d(fx_{n-1}, Tx_n, u), d(fx_n, Tx_{n-1}, u)\}$$

$$= max\{d(y_{n-1}, y_{n+1}, u), d(y_n, y_n, u)\}$$

$$= d(y_{n-1}, y_{n+1}, u)$$

Using (2.4) and (2.9), the inequality (2.10) gives

$$d_n(u) \le a\, d_{n-1}(u) + c\, d(y_{n-1}, y_{n+1}, u)$$

$$\le a\, d_{n-2}(u) + c\,[2 - c(1-h)]d_{n-2}(u)$$

$$= [1 - c^2(1-h)]d_{n-2}(u)$$

Hence

$$d_n(u) \le [1 - \beta^2(1-h)]d_{n-2}(u)$$

Proceeding in this manner, we obtain

(2.11) $$d_n(u) \le \left(1 - \beta^2(1-h)\right)^{\left[\frac{n}{2}\right]} d_0(u)$$

where $\left[\frac{n}{2}\right]$ stands for the greatest integer not exceeding $\frac{n}{2}$. Since $\beta = inf\{c(x,y): x, y \in X\} > 0$ and $h \in (0,1)$, which implies that

(2.12) $$\lim_{n\to\infty} d_n(u) = 0.$$

Now, we prove that $\{y_n\}$ is Cauchy. Suppose to the contrary, that $\{y_n\}$ is not a Cauchy sequence in X. Then for every $\epsilon > 0$, there exists $u \in X$ and strictly increasing sequences $\{m_k\}, \{n_k\}$ of positive integers such that $m_k > n_k \ge k$ with

(2.13) $$d(y_{m_k}, y_{n_k}, u) \ge \epsilon$$

Without loss of generality, we can suppose that also

(2.14) $$m_k > n_k \ge k, \quad d(y_{m_k}, y_{n_k}, u) \ge \epsilon, \quad d(y_{n_k}, y_{m_{k-2}}, u) < \epsilon$$

From (2.14) and the tetrahedral inequality (that holds for a 2-metric space), we have

(2.15) $$\epsilon \leq d(y_{m_k}, y_{n_k}, u)$$

$$\leq d(y_{m_k}, y_{n_k}, y_{m_k-2}) + d(y_{m_k}, y_{m_k-2}, u) + d(y_{m_k-2}, y_{n_k}, u)$$

$$\leq d(y_{m_k-2}, y_{n_k}, u) + d(y_{m_k}, y_{n_k}, y_{m_k-2}) + d(y_{m_k-1}, y_{m_k-2}, u)$$

$$+ d(y_{m_k}, y_{m_k-1}, u) + d(y_{m_k}, y_{m_k-2}, y_{m_k-1})$$

$$\leq \epsilon + d(y_{m_k}, y_{n_k}, y_{m_k-2}) + d_{m_k-2}(u) + d_{m_k-1}(u) + d_{m_k-2}(y_{m_k})$$

On letting $k \to +\infty$ in (2.15) and using (2.2), (2.7), (2.12), we get

(2.16) $$lim_{k \to +\infty} d(y_{m_k}, y_{n_k}, u) = \epsilon$$

It follows from (2.14) that

$$0 < d(y_{n_k}, y_{m_k}, u) - d(y_{n_k}, y_{m_k-2}, u)$$

$$\leq d(y_{n_k}, y_{m_k-2}, u) + d(y_{m_k-2}, y_{m_k}, u) + d(y_{n_k}, y_{m_k}, y_{m_k-2}) - d(y_{n_k}, y_{m_k-2}, u)$$

$$= d(y_{m_k-2}, y_{m_k}, u) + d(y_{n_k}, y_{m_k}, y_{m_k-2})$$

$$\leq d_{m_k-2}(y_{m_k}) + d_{m_k-2}(u) + d_{m_k-1}(u) + d(y_{n_k}, y_{m_k}, y_{m_k-2})$$

On making $k \to +\infty$, we immediately obtain that:

(2.17) $$lim_{k \to +\infty} d(y_{n_k}, y_{m_k-2}, u) = \epsilon$$

Note that

$$|d(y_{n_k}, y_{m_k-1}, u) - d(y_{n_k}, y_{m_k}, u)| \leq d_{m_k-1}(u) + d_{m_k-1}(y_{n_k})$$

$$|d(y_{n_k+1}, y_{m_k}, u) - d(y_{n_k}, y_{m_k}, u)| \leq d_{n_k}(u) + d_{n_k}(y_{m_k})$$

$$|d(y_{n_k+1}, y_{m_k-1}, u) - d(y_{n_k}, y_{m_k}, u)| \leq d_{n_k}(u) + d_{m_k-1}(y_{n_k})$$

$$+ d_{m_k-1}(u) + d_{n_k}(y_{m_k-1})$$

On letting $k \to +\infty$, in these inequalities and by using inequalities (2.2), (2.7), (2.12), (2.15) and (2.17), we obtain;

(2.18) $$\lim_{k\to+\infty} d(y_{n_k}, y_{m_k-1}, u) = \epsilon, \quad \lim_{k\to+\infty} d(y_{n_k+1}, y_{m_k}, u) = \epsilon,$$

$$\lim_{k\to+\infty} d(y_{n_k+1}, y_{m_k-1}, u) = \epsilon.$$

Now, using (1.5), we have

(2.19) $$d(y_{m_k}, y_{n_k+1}, u) = d(Tx_{m_k-1}, Tx_{n_k}, u)$$

$$\leq a\,max\{d(fx_{m_k-1}, fx_{n_k}, u), d(fx_{m_k-1}, Tx_{m_k-1}, u), d(fx_{n_k}, Tx_{n_k}, u)$$

$$, \tfrac{1}{2}[M(x_{m_k-1}, x_{n_k}, u) + m(x_{m_k-1}, x_{n_k}, u)]\}$$

$$+c[M(x_{m_k-1}, x_{n_k}, u) + h\,m(x_{m_k-1}, x_{n_k}, u)]$$

$$= a\,max\{d(y_{m_k-1}, y_{n_k}, u), d(y_{m_k-1}, y_{m_k}, u), d(y_{n_k}, y_{n_k+1}, u)$$

$$, \tfrac{1}{2}[M(x_{m_k-1}, x_{n_k}, u) + m(x_{m_k-1}, x_{n_k}, u)]\}$$

$$+c[M(x_{m_k-1}, x_{n_k}, u) + h\,m(x_{m_k-1}, x_{n_k}, u)]$$

where a and c are evaluated at (x_{m_k-1}, x_{n_k}) and

$$m(x_{m_k-1}, x_{n_k}, u) = min\{d(fx_{m_k-1}, Tx_{n_k}, u), d(fx_{n_k}, Tx_{m_k-1}, u)\}$$

$$= min\{d(y_{m_k-1}, y_{m_k+1}, u), d(y_{n_k}, y_{m_k}, u)\}$$

$$M(x_{m_k-1}, x_{n_k}, u) = max\{d(fx_{m_k-1}, Tx_{n_k}, u), d(fx_{n_k}, Tx_{m_k-1}, u)\}$$

$$= max\{d(y_{m_k-1}, y_{m_k+1}, u), d(y_{n_k}, y_{m_k}, u)\}$$

On letting $k \to +\infty$ in (2.19), using (2.12), (2.16) and (2.18), we have

$$\epsilon \leq a\,max\left\{\epsilon, 0, 0, \tfrac{1}{2}[\epsilon + \epsilon]\right\} + c[\epsilon + h\epsilon]$$

$$= (a + c(1 + h))\epsilon$$

$$= \big(1 - c(1-h)\big)\epsilon$$

This is a contradiction, since $\beta = inf\{c(x,y): x,y \in X\} > 0$ and $h \in (0,1)$. Thus, our supposition was wrong and therefore, $\{y_n\}$ is a Cauchy Sequence in X.

For cases (a) and (b) suppose that X is complete. Then Cauchy sequence $\{y_n\}$ will converge to a point p in X and then $fx_n \to p$ and $Tx_n \to p$ as $n \to +\infty$.

Case (a): Suppose that f is surjective. Then there exists a point z in X such that $p = fz$.

From (1.5), we have

(2.20) $d(fz,Tz,u) \le d(fz,Tz,y_{n+1}) + d(fz,y_{n+1},u) + d(y_{n+1},Tz,u)$

$$= d(fz,y_{n+1},u) + d(fz,Tz,y_{n+1}) + d(Tx_n,Tz,u)$$

$$\le d(fz,y_{n+1},u) + d(fz,Tz,y_{n+1})$$

$$+a\ max\left\{d(fx_n,fz,u), d(fx_n,Tx_n,u), d(fz,Tz,u), \tfrac{1}{2}[M(x_n,z,u) + m(x_n,z,u)]\right\}$$

$$+c[M(x_n,z,u) + h\ m(x_n,z,u)]$$

Since

$$\lim_{n\to+\infty} M(x_n,z,u) = \lim_{n\to+\infty} \max\{d(fx_n,Tz,u),(fz,Tx_n,u)\}$$

$$= d(fz,Tz,u)$$

and

$$\lim_{n\to+\infty} m(x_n,z,u) = \lim_{n\to+\infty} \min\{d(fx_n,Tz,u),(fz,Tx_n,u)\}$$

$$= 0.$$

Taking limit $n \to +\infty$ in the inequality (2.20), we have

$$d(fz,Tz,u) \le \sup_{x,y\in X}(a+c)d(fz,Tz,u) < d(fz,Tz,u)$$

implies that $fz = Tz = p$.

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Case (b): Suppose f is continuous. Then since $\lim_{n\to+\infty} y_n = p$, we have $\lim_{n\to+\infty} fy_n = fp$ and then $\lim_{n\to+\infty} ffx_n = fp$. Also f and T are compatible and $\lim_{n\to+\infty} fx_n = \lim_{n\to+\infty} Tx_n = \lim_{n\to+\infty} y_{n+1} = p$. Hence

(2.21) $$\lim_{n\to+\infty} d(fTx_n, Tfx_n, u) = 0$$

Note that

$$d(ffx_n, Tfx_n, u) \le d(ffx_n, Tfx_n, fTx_n) + d(ffx_n, fTx_n, u) + d(fTx_n, Tfx_n, u)$$

$$= d(ffx_n, Tfx_n, fTx_n) + d(ffx_n, ffx_{n+1}, u) + d(fTx_n, Tfx_n, u)$$

On taking limit $n \to +\infty$ and using (2.21), we have $\lim_{n\to+\infty} d(ffx_n, Tfx_n, u) = 0$. Since $\lim_{n\to+\infty} ffx_n = fp$, it follows that $\lim_{n\to+\infty} Tfx_n = fp$.

Applying (1.5) again, we have

(2.22) $$d(fp, Tp, u) \le d(fp, Tp, fy_{n+1}) + d(fp, fy_{n+1}, u) + d(fy_{n+1}, Tp, u)$$

$$\le d(fp, Tp, fy_{n+1}) + d(fp, fy_{n+1}, u) + d(Tp, fTx_n, u)$$

$$\le d(fp, Tp, fy_{n+1}) + d(fp, fy_{n+1}, u) + d(Tp, fTx_n, Tfx_n)$$

$$+ d(Tp, Tfx_n, u) + d(Tfx_n, fTx_n, u)$$

$$\le d(fp, Tp, fy_{n+1}) + d(fp, fy_{n+1}, u) + d(Tp, fTx_n, Tfx_n)$$

$$+ d(Tfx_n, fTx_n, u) + a\, max\{d(fp, ffx_n, u), d(fp, Tp, u), d(ffx_n, Tfx_n, u)$$

$$, \tfrac{1}{2}[M(p, fx_n, u) + m(p, fx_n, u)]\} + c[M(p, fx_n, u) + h\, m(p, fx_n, u)]$$

Note that

$$\lim_{n\to+\infty} M(p, fx_n, u) = \lim_{n\to+\infty} \max\{d(fp, Tfx_n, u), (ffx_n, Tp, u)\}$$

$$= \lim_{n\to+\infty} \max\{d(fp, Tfx_n, u), (ffx_n, Tp, u)\}$$

$$= d(fp, Tp, u)$$

and

$$\lim_{n\to+\infty} m(p, fx_n, u) = \lim_{n\to+\infty} \min\{d(fp, Tfx_n, u), (ffx_n, Tp, u)\}$$

$$= \lim_{n\to+\infty} \min\{d(fp, Tfx_n, u), (ffx_n, Tp, u)\}$$

$$= 0.$$

On letting $n \to +\infty$ in the inequality (2.22), we have

(2.23) $$d(fp, Tp, u) \leq \sup_{x,y\in X} (a + c)\, d(fp, Tp, u)$$

implies that $fp = Tp$.

Case (c): In this case $p \in f(X)$. Let $z \in f^{-1}(p)$. Then $p = f z$ and the proof is complete by case (a).

Case (d): In this case $p \in T(X) \subseteq f(X)$ and the proof is complete by case (c).

Finally, we shall prove that f and T have at most one coincidence point. On the contrary, suppose that f and T have two coincidence points p and q. Then from (1.5) with a and b evaluated at (p, q), we have

(2.24) $$d(Tp, Tq, u) \leq a\, max \left\{d(fp, fq, u), d(fp, Tp, u), d(fq, Tq, u), \tfrac{1}{2}[M(p, q, u) + m(p, q, u)]\right\}$$

$$+c[M(p, q, u) + h\, m(p, q, u)]$$

$$= [a + c(1 + h)]\, d(Tp, Tq, u)$$

because

$$M(p, q, u) = max\{d(fp, Tq, u), d(fq, Tp, u)\}$$

$$= d(Tq, Tp, u)$$

$$m(p, q, u) = min\{d(fp, Tq, u), d(fq, Tp, u)\}$$

$$= d(Tq, Tp, u)$$

Hence by (1.7),

(2.25) $$(Tp, Tq, u) \leq [1 - c(1 - h)]\, d(Tp, Tq, u)$$

implying $Tp = Tq$ by (1.6) and hence $fp = fq$.

Corollary 2.2 Let (X, d) be a complete 2-metric space and T is self mapping of X satisfying (1.5) with $f = I$, the identity map on X, where $h = 1$, a and b satisfying (1.6) and (1.7). Then T has a unique fixed point and at this fixed point T is continuous.

Proof The existence and uniqueness of the fixed point comes from Theorem 2.1 by setting $f = I$. To prove continuity, let $\{y_n\} \subset X$ with $\lim_{n\to+\infty} y_n = p$, p the unique fixed point of T.

We apply inequality (1.6), where a, c are evaluating at (y_n, p).

(2.26) $$d(Ty_n, Tp, u) \leq a\ max\left\{d(y_n, p, u), d(y_n, Ty_n, u), d(p, Tp, u), \tfrac{1}{2}[M(y_n, p, u) + m(y_n, p, u)]\right\}$$

$$+c[M(y_n, p, u) + m(y_n, p, u)]$$

$$\leq a\left(d(y_n, p, u) + d(p, Ty_n, u)\right) + c[d(p, Ty_n, u) + d(y_n, p, u)]$$

$$= (a + c)\ d(y_n, p, u) + (a + c) d(p, Ty_n, u)$$

$$= (1 - c)\ d(y_n, p, u) + (1 - c) d(p, Ty_n, u)$$

Hence

(2.27) $$d(Ty_n, Tp, u) \leq (1 - \beta)\ d(y_n, p, u) + (1 - \beta) d(p, Ty_n, u)$$

Since $\beta = inf\{c(x, y) : x, y \in X\} > 0$. Hence we get

(2.28) $$d(Ty_n, Tp, u) \leq \left(\frac{1}{\beta} - 1\right)\ d(y_n, p, u)$$

Taking the limit as $n \to +\infty$ yields

$$\lim_{n\to+\infty} Ty_n = Tp.$$

Therefore T is continuous at p.

CONFLICT OF INTEREST

No conflict of interest was declared by the authors.

AUTHOR'S CONTRIBUTIONS

All authors contributed equally and significantly to writing this paper. All authors read and approved the final manuscript.

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