Finite Element Formulation for Large Deformation Problem

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ABSTRACT

In this paper finite element procedure has been applied to solve the large deformation and large strain problems. Linear and nonlinear mechanics followed by finite element formulation has been used to solve the truss problem. The aim of this research paper is to give an overview about the algorithm used to solve the complex problem. For that the behaviour of the hyperelastic material is studied subjected to large strain by using the code FLagSHyP (Finite Element Large Strain Hyperelastic Program) written by Javier Bonet and Richard D. Wood. The result obtained from the code FLagSHyP is also compared from the result obtained by the abaqus software. In this Work three hyperelastic constitutive models i.e. Odgen model, Incompressible neo-hookean model and Mooney-Rivlin model have been used to solve the truss problem subjected to large strain.

Keywords: Finite Element,Hyperelastic program,Large deformation

I INTRODUCTION

In linear and non-linear finite element analysis involving large displacements, large strains and material non-linearities, it is necessary to resort to an incremental formulation of the equations of motion. Various formulations are used in practice. Some procedures are general and others are restricted to account for material non-linearities only, or for large displacements but not for large strains, or the formulation may only be applicable to certain types of elements. Limited results have been obtained in dynamic non-linear analysis involving large displacements and large strains.

The earliest non-linear finite element analysis were essentially based on extensions of linear analysis and have been developed for specific applications. The procedures were primarily developed on an intuitive basis in order to obtain solutions to the specific problems considered. However, to provide general analysis capabilities using isoparametric (and related) elements a general formulation need to be used. The isoparametric finite element discretization procedure has proved to be very effective in many applications, and lately it has been shown that general nonlinear formulations based on principles of continuum mechanics can be efficiently implemented.

Now a day, analysis of non-linear behavior of components is most likely to involve a simulation, because of the easy availability of finite element software's to provide good results with accuracy it is necessary to understand the fundamental of linear and non-linear continuum mechanics, non-linear finite element formulation and solution procedure. A number of books are available to provide background on this subject, for example the book of JN Reddy [1], Klaus Hackl and Mehndi goodarazi [2] explain the basic concepts of linear and non-linear continuum mechanics (stress and tensor, elasticity, Hooke's law, displacement functions. The book of Javier Bonet and Richard D. Wood [3], Morton E. Gurtin [4], J. Tensley Oden [5], explain the concepts dealing with different types of non-linearities viz geometrical non-linearity, material non-linearity and contact non-linearity; kinematics concepts viz deformation tensors, strains; kinetics concepts viz Cauchy stress, first and second piola kirchoff stress and different governing equations involved in it.

Klaus-Jurgen Bathe et al. [6] derived and compared the two-finite element incremental formulations viz Updated lagrangian formulation and total lagrangian formulation for nonlinear static and dynamic analysis.

Concerning the application of finite element methods to the solution of problem involving large displacement and rotation we note that there are basically two different approaches used in FEM first is the total lagrangian formulation and the second is updated lagrangian formulation.

The formulation includes large displacement, large strain and material non linearities. The formulations are listed below

Total lagrangian formulation:

$$\int_{0_v}^{t} {}_0^tC_{ijrs}\, {}_0^t\varepsilon_{rs}\,\delta\, {}_0^t\varepsilon_{ij}\, {}_t^0dv + \int_{0_v}^{t} {}_0^tS_{ij}\,\delta\, {}_0^t\eta_{ij}\, {}_t^0dv = {}^{t+\Delta t}_{\ \ 0}R - \int_{0_v}^{t} {}_0^tS_{ij}\,\delta\, {}_0^te_{ij}\, {}_t^0dv \quad (1)$$

Updated lagrangian formulation:

$$\int_{t_v} {}^{t+\Delta t}_{\ \ t}C_{ijrs}\, {}^{t+\Delta t}_{\ \ t}\varepsilon_{rs}\,\delta\, {}^{t+\Delta t}_{\ \ t}\varepsilon_{ij}\, {}_0^tdv + \int_{t}^{t+\Delta t} {}_0^t\zeta_{ij}\,\delta\, {}^{t+\Delta t}_{\ \ t}\eta_{ij}\, {}_t^0dv = {}^{t+\Delta t}_{\ \ 0}R - \int_{t_v}^{t+\Delta t} {}_0^t\zeta_{ij}\,\delta\, {}^{t+\Delta t}_{\ \ t}e_{ij}\, {}_t^0dv \quad (2)$$

Where $0, t\ and\ t+\Delta t$ are time step/load level showing the reference configuration, new reference configuration and current configuration respectively; ${}_0^tC_{ijrs}$, ${}^{t+\Delta t}_{\ \ t}C_{ijrs}$ are the components of constitutive tensor at time t referred to the configuration at time 0 and t respectively; ${}_0^t\varepsilon_{rs}$, ${}^{t+\Delta t}_{\ \ t}\varepsilon_{rs}$ components of strain increment tensor referred to configuration at time 0 and t respectively; ${}_0^te_{ij}$, ${}^{t+\Delta t}_{\ \ t}e_{ij}$ are linear part of strain increment at time 0 and t respectively; ${}_0^t\eta_{ij}$, ${}^{t+\Delta t}_{\ \ t}\eta_{ij}$ are non linear part of strain imcrement at time 0 and t respectively; ${}_0^tS_{ij}$ are the components of second piola kirchoff stress tensor at time t referred to configuration at time 0; ${}_0^t\zeta_{ij}$ are the components of Cauchy stress tencor at time t and ${}^{t+\Delta t}_{\ \ 0}R$ is the external virtual work expression corresponding to configuration at time $t+\Delta t$.

1.1Finite element procedure

In this procedure the problem is solved in incremental steps. It uses the iterative methods to give the results. The steps involved in the procedure.

1.1.1 Deformation Gradient tensor

The deformation gradient tensor is the second order tensor which maps the line elements in the reference configuration to the line elements in the current configuration

$$\mathbf{F=dx/dX} \qquad \mathbf{(3)}$$

WheredX is the line element emanating from position X in the reference configuration and dx is the line element in the current configuration

1.1.2 Strain

Strain (Ɛ) is a measure of deformation representing the displacement between particles in the body relative to a reference length. A general deformation of a body can be expressed in the form x=F(X) where X is the reference position of material points in the body and I is the identity tensor

$$\mathcal{E} = \frac{\vartheta(\mathbf{x}-\mathbf{X})}{\vartheta \mathbf{x}} = \mathbf{F}' - \mathbf{I} \qquad \mathbf{(4)}$$

1.1.3 Cauchy stress tensor

The Cauchy stress tensor (σ) is a second order tensor that completely defines the state of stress at a point inside a material in the deformed state. The tensor relates a unit-length direction vector n to the stress vector T^n across an imaginary surface perpendicular to$n$ and described as

$$\mathbf{T_j^n = \sigma_{ij}} \qquad \mathbf{i, j = 1, 2, 3(5)}$$

1.1.4 Principle of Virtual Work

Consider the motion of a body, the principle of virtual work is used to express the equilibrium of the body

$$\int_{s_t} \boldsymbol{\delta u^T T ds} + \int_{v} \boldsymbol{\delta u^T f dv} = \int \boldsymbol{\delta \mathcal{E}^T \sigma dv} \mathbf{(6)}$$

Whereare $\int_{s_t} \delta u^T T ds + \int_{v} \delta u^T f dv$ internal virtual work due to traction and body force andis an external virtual work $\int \delta \mathcal{E}^T \sigma dv$

1.1.5For linear and nonlinearproblem

The solution cannot be obtaineddirectly as above equation give the solution in discrete time step so we use the lagrangian formulation [1] to solve the problem which integrate the unknown variables (displacement and strain) with reference to initial position or first position. If there is nonlinearproblem, then it became necessary to linearize it after the lagrangian formulation

1.2 FORMULATION OF CONTINUUM MECHANICS INCREMENTAL EQUATION

Consider the motion of a body in a Cartesian co-ordinate system. The aim is to evaluate the equilibrium positions of the body at the discrete time points 0,Δt,2Δt,3Δt…where Δt is an increment in time. Assume that the solution for the kinematic and static variables for all time steps from time 0 to time t, inclusive, have been solved, and that the solution for time t+Δt is required next. It is noted that the solution process for the next required equilibrium position is typical and would be applied repetitively until the complete solution path has been solved.

Since the solution is known at all discrete points $0, \Delta t, 2\Delta t \ldots\ldots t$, the basic aim of the formulation is to generate an equation of virtual work from which the unknown static and kinematic variables in the configuration at time $t + \Delta t$ can be solved. The principle of virtual work is used to express the equilibrium of the body in the configuration at the time $+\Delta t$. It is stated as

$$\int_{t+\Delta t_v}^{0} {}^{t+\Delta t}_{0}\sigma_{ij}\, \delta\, {}^{0}_{t+\Delta t}e_{ij}\, {}^{t+\Delta t}_{0}dv = {}^{t+\Delta t}_{0}R \quad (7)$$

Where ${}^{t+\Delta t}_{0}R$ is the external virtual work expression and shown as,

$${}^{t+\Delta t}_{0}R = \int_{t+\Delta t_v}^{0} {}^{t+\Delta t}_{t+\Delta t}f^0_k\, \delta u^0_k\, {}^{t+\Delta t}_{0}dv + \int_{t+\Delta t_a}^{0} {}^{t+\Delta t}_{t+\Delta t}t^0_k\, \delta u_k\, {}^{t+\Delta t}_{0}dv \qquad k = 1, 2, 3 \quad (8)$$

Where ${}^{t+\Delta t}_{0}f^0_k$, ${}^{t+\Delta t}_{t+\Delta t}t^0_k$ are the body force vector and surface traction vector in the configuration at the time $t + \Delta t$ respectively, δu^0_k is the (virtual) variation in the current displacement component.

The difficulty here being that configuration of the body at time $t + \Delta t$ is unknown so the above equation cannot be solved directly. A solution can be obtained by referring all the variables to a known previously calculated equilibrium configuration for that two approaches have been used i.e. Total lagrangian formulation and Updated lagrangian formulation

Truss Problem

A four noded truss of dimension 140mm ×1.414mm ×0.707mm has been chosen. The initial orientation angle is 45 degree. There is a hinge support on node 1 and roller support on node 2 which constrained the motion in x direction and allows to move in y direction as shown in Fig.1(all dimension in mm). The displacement of the top node is prescribed downward in increment of 1mm.

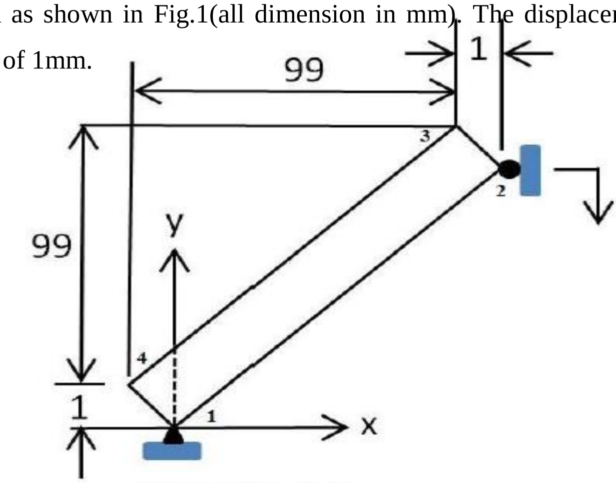

Fig.1 Truss with prescribed displacement in downward direction

The problem is solved using 198 increments (1 mm increment in every increment step). A truss is divided into 700 elements as shown in Fig.2 The convergence criterion is set to 1.e-10 in the code with a maximum 25 iterations in every step.

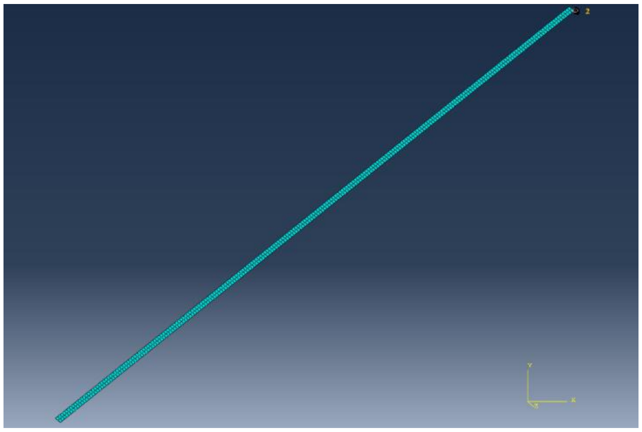

Fig2. Meshed Geometry with node where analysis is performed

II RESULT

The vertical Reaction vs. displacement graph of node 2 obtained from FLagSHyP code is shown in Fig.3

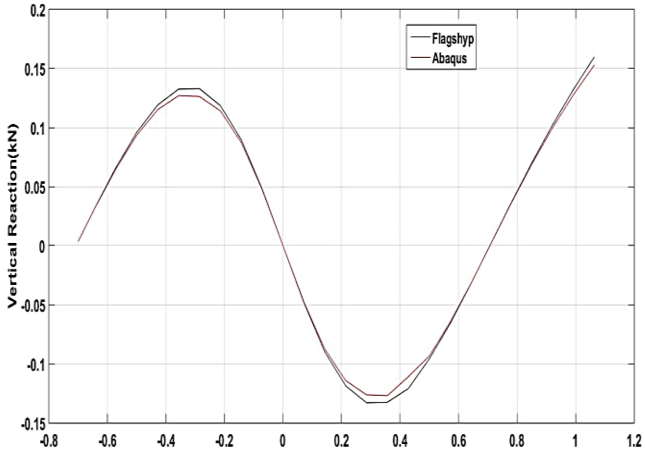

Fig.3 Vertical Reaction v/s Displacement of node 2

In Fig.3, x is the vertical position of node 2 in every increment step and L is the length of the truss. It can be observed that the truss exhibit non-linear snap through behaviour (also see Fig.4). In this non-linear instability region the equilibrium path goes from one stable point (at which force is 0.1322KN) to another stable point(at which force is 0.1329 KN).

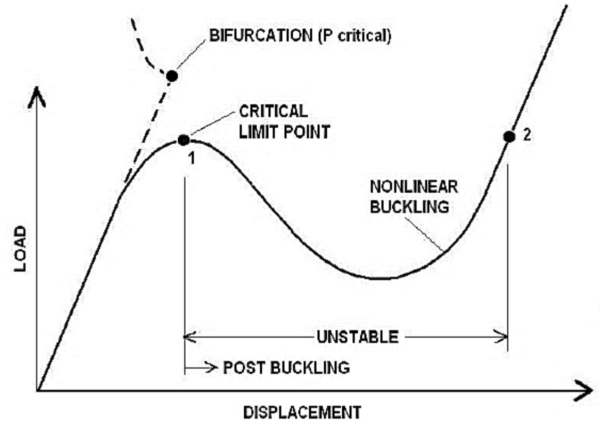

Fig.4 Non-Linear Snap through behaviour of Truss [7]

III CONCLUSION

In the work, the behaviour of hyperelastic material (incompressible elastomer) has been studied for large strains through the Finite element large strain hyper elasticity program (FLagSHyP) written by Javier Bonet and Richard D. Wood. The results have been scrutinized to describe the behaviour of the material in different boundary and load conditions. The following conclusions are illustrated in this work.

1. The three hyperelastic constitutive model odgen, neo-hookean and mooney-rivlin are used for the analysis of truss problem, plate with hole problem and pressurized cylinder problem.
2. Only material non-linearity is considered as the program is structured using Updated lagrangian formulation.
3. The results depict that the material is highly nonlinear elastic.
4. The non-linear snap through behaviour is obtained in truss problem
5. In plate with hole problem it is that intensity of stresses are high near the hole because of the stress concentration in this region whereas the stresses are minimum on the edge of the plate because forces are evenly distributed here.

This work can be extended to viscoelastic materials and elasto-plastic materials. The software on total lagrangian method can be developed and results compared with the updated lagrangean method.

REFRENCES

[1] J.N. Reddy., (2008). *An Introduction to Continuum Mechanics*, first-edition, Cambridge university press, New York.

[2] Klaus Hackl., Mehdi Goodarzi., (2010). *An Introduction to linear Continuum Mechanics*, lecture notes, Ruhr University Press, Bochum (Germany).

[3] Javier Bonet., Richard D. Wood., (1997). *Non-Linear Continuum Mechanics for Finite Element Analysis*, Cambridge University Press, New York.

[4] Morton E. Gurtin., (1982). *An Introduction to Continuum Mechanics*, volume 158 first- Edition, Academic Press Inc, London.

[5] J. Tinsley Oden., (2006). *Finite Element of Non-Linear Continua*, Dover Publication Inc, New York

[6] Klaus-Jurgen Bathe., Edward L. Wilson., (1975). *Finite Element Formulations for Large Strain Deformation Dynamic Analysis*, International Journal for Numerical Method in Engineering, Vol:9, 353-386.

[7] Rubber Industry Report (2014), http://www.rubberworld.com/RWmarket_report.asp?id=333