# FIXED POINT THEOREMS FOR *F*-EXPANDING MAPPINGS OF G-METRIC SPACE

**Mr. Shaikh Mohammed Sirajuddin Mohammed Salimuddin**

*Research Scholar (In Mathematics), OPJS University ,Churu, Rajashtan(India)*

## ABSTRACT

*Introduced Fixed point theorem is define new concept of* F*-contraction mapping and which generalizes the Banach Space contraction principle, we present some new fixed point results for* F*-expanding mappings, especially on a complete* G*-metric space.*

***Keywords:* *Fixed Point* F*-Contraction Map,* F*-Expanding Map,* G*-Metric Space***

## 1 INTRODUCTION

Let (X,d)(X,d) be a metric space. A mapping T:X→X is said to be expanding if

$$\forall x,y\in X \quad d(Tx,Ty) \geq \lambda d(x,y), \text{where} \lambda>1.$$

The condition $\lambda>1$ is important, the function T:R→R defined by $Tx=x+e^x$ satisfies the condition $|Tx-Ty| \geq |x-y|$ for all x,y∈R, and *T* has no fixed point.

For an expanding map, the following result is well known.

### Theorem 1.1

*Let* (X,d)(X,d) *be a complete metric space, and let* T:X→X *be surjective and expanding. Then T is bijective and has a unique fixed point.*

It follows from the Banach contraction principle and the following very simple observation.

### Lemma 1.2

*If* T:X→X *is surjective, then there exists a mapping* $T^*$:X→X *such that* $T\circ T^*$ *is the identity map on X.*

**Proof**

For any point x∈X, let yx∈X be any point such that Tyx= x. Let $T^*x=y_x$ for all x∈X Then $(T\circ T^*)(x)=T(T^*x)$ for all x∈X .

In the present paper, we introduce a new type of expanding mappings.

### Definition 1.3

Let F be the family of all function F:(0,+∞)→R such that

*(F1): F* is strictly increasing, i.e., for all α,β∈(0,+∞), if α< β, then F(α)<F(β);

*(F2):* for each sequence $\{\alpha_n\} \subset (0,+\infty)$ , the following holds:

$\lim_{n\to\infty} \alpha n = 0$ if and only if $\lim_{n\to\infty} F(\alpha n) = -\infty$

*(F3):* there exists k∈(0,1) such that $\lim_{n\to 0+} \alpha k F(\alpha) = 0$

ISBN: 978-93-86171-46-7

**Definition 1.4**

Let (X,d) be a metric space. A mapping $T:X\to X$ is called $F$-expanding if there exist $F\in F$ and $t > 0$ such that for all $x,y\in X$,

$d(x,y)>0 \Rightarrow F(d(Tx,Ty)) \geq F(d(x,y)) + t.$ (2)

When we consider in (2) the different types of the mapping F∈FF∈F, then we obtain a variety of expanding mappings.

**Example 1.5**

Let $F1(\alpha)=\ln\alpha$ .It is clear that F1F1 satisfies ($F$1), ($F$2), ($F$3) for any $k\in(0,1)$. Each mapping $T:X\to X$ satisfying (2) is an F1-expanding map such that

$d(Tx,Ty) \geq e^t d(x,y)$ for all $x,y \in X$, .

It is clear that for $x,y \in X$ such that $x=y$, the inequality $d(Tx,Ty) \geq e^t d(x,y)$ also holds.

**Example 1.6**

If $F2(\alpha)=\ln \alpha+\alpha$, $\alpha>0$, then F1 satisfies ($F$1), ($F$2) and ($F$3), and condition (2) is of the form

$d(Tx,Ty)ed^{(Tx,Ty)-d(x,y)} \geq e^t d(x,y)$ for all $x,y\in X$..

**Example 1.7**

Consider $F3(\alpha)=\ln(\alpha^2+\alpha)$, $\alpha>0$. F3 satisfies ($F$1), ($F$2) and ($F$3), and for F3-expanding $T$, the following condition holds:

$$d(Tx,Ty).\ \frac{d(Tx,Ty)+1}{d(x,y)+1} \geq e^t d(x,y) \text{ for all } x,y \in X.$$

**Example 1.8**

Consider $F4(\alpha)=\arctan(-\frac{1}{(\alpha)})$, $\alpha>0\alpha>0$. F4 satisfies ($F$1), ($F$2) and ($F$3), and for F4-expanding $T$, the following condition holds:

$$d(Tx,Ty) \geq \left(\frac{1+\frac{\tan t}{d(x,y)}}{1-\tan t \, .d(x,y)}\right) d(x,y) \text{ for some } 0 < t < \frac{\pi}{2}$$

Here, we have obtained a special type of nonlinear expanding map

$$d(Tx,Ty) \geq \varphi\,(d(x,y))d(x,y).$$

Other functions belonging to F are, for example, $F(\alpha)=\ln(\alpha^n)$, $n\in N$, , $\alpha > 0$;

$F(\alpha)=\ln(\arctan \alpha)$, $\alpha > 0$.

Now we recall the following.

**Definition 1.9**

Let (X,d) be a metric space. A mapping $T:X\to X$ is an $F$-contraction on $X$ if there exist $F\in F$ and $t > 0$ such that for all $x, y \in X$,

$d(Tx,Ty)>0 \Rightarrow t + F(d(Tx,Ty)) \leq F(d(x,y))$.. (3)

For such mappings, Wardowski [1] proved the following theorem.

 ISBN: 978-93-86171-46-7

**Theorem 1.10**

*Let* (X,d) *be a complete metric space and* $T:X\to X$ *be an F-contraction. Then T has a unique fixed point* $u \in X$ *and for every* $x\in X$, *a sequence* $\{x_n = T^n x\}$ *is convergent to u.*

## II THE RESULT

In this section, we give some fixed point theorem for *F*-expanding maps.

**Theorem 2.1**

*Let* (X,d) *be a complete metric space and* $T:X\to X$ *be surjective and F-expanding. Then T has a unique fixed point.*

**Proof**

From Lemma 1.2, there exists a mapping $T^*:X\to X$ such that $T\circ T$ is the identity mapping on *X*.

Let $x,y \in X$ be arbitrary points such that $x\neq y$, and let $z=T^*x$ and $w=T^*y$ (obviously, $z\neq w$).

By using (2) applied to *z* and *w*, we have

$F(d(Tz,Tw)) \geq F(d(z,w)) + t.$

Since $Tz=T(T^*x) = x$ and $Tw=T(T^*y)=y$, then

$F(d(x,y)) \geq F(d(T^*x,T^*y)) + t,$

so $T^*:X\to X$ is an *F*-contraction. By Theorem 1.10, $T^*$ has a unique fixed point $u\in X$. In particular, *u* is also a fixed point of *T* because $T^*u = u$ implies that $Tu=T(T^*u) = u$.

Let us observe that *T* has at most one fixed point. If $u,v \in X$ and $Tu=u\neq v$, then we would get the contradiction

$F(d(Tu,Tv)) \geq F(d(u,v)) + t,$

$0=F(d(Tu,Tv)) - F(d(u,v)) \geq t > 0,$

so the fixed point of *T* is unique.

**Remark 2.2**

If *T* is not surjective, the previous result is false. For example, let $X=[0,\infty)$ endowed with the metric $d(x,y) = |x-y|$ for all $x,y \in X$, and let $T:X\to X$ be defined by $Tx=2x + 1$ for all $x \in X$ Then *T* satisfies the condition $d(Tx,Ty) \geq 2d(x,y)$ for all $x,y \in X$ and *T* is fixed point free.

## III APPLICATIONS TO *G*-METRIC SPACES

In 2006 Mustafa and Sims (see [2] and the references therein) introduced the notion of a *G*-metric space and investigated the topology of such spaces. The *G*-metric space is as follows.

**Definition 3.1**

Let *X* be a nonempty set. A function $G:X\times X\times X\to[0,\infty)$ satisfying the following axioms:

(*G1*) $G(x,y,z) = 0$ if $x=y=z$,

(*G2*) $G(x,x,y) > 0$ for all $x, y \in X$ with $x \neq y$,

(*G3*) $G(x,x,y) \leq G(x,y,z)$ for all $x,y,z\in X$ with $z \neq y$,

(*G4*) $G(x,y,z) \leq G(x,z,y) = G(y,z,x)=$ (symmetry in all three variables),

 ISBN: 978-93-86171-46-7

(G5) $G(x,y,z) \le G(x,a,a) + G(a,y,z)$ for all $x, y, z, a \in X$,

is called a *G*-metric on *X*, and the pair (X,G) is called a *G*-metric space.

Recently, Samet et al. [3] observed that some fixed point theorems in the context of *G*-metric spaces can be concluded from existence results in the setting of quasi-metric spaces. Especially, the following theorem is a simple consequence of Theorem 1.10.

**Theorem 3.2**

*Let* (X,G) *be a complete G-metric space, and let* $T:X\to X$ *satisfy one of the following conditions*:

*(a) T is an F-contraction of type I on a G-metric space X, i.e., there exist*
$F\in \mathcal{F}$ *and* t>0t>0 *such that for all* $x,y \in X$,

$$G(Tx,Ty,Ty) > 0 \Rightarrow t + F(G(Tx,Ty,Ty)) \le F(G(x,y,y)); \quad (4)$$

*(b) T is an F-contraction of type II on a G-metric space X, i.e., ther exist* $F\in\mathcal{F}$
*and* t>0 *such that for all* $x, y, z \in X$,

$$G(Tx,Ty,Tz) > 0 \Rightarrow t + F(G(Tx,Ty,Tz)) \le F(G(x,y,z)). \quad (5)$$

*Then T has a unique fixed point* $u\in X$, *and for any* $x\in X$ , *a sequence* $\{ x_n=T^n x \}$ *is G-convergent to u*.

The previous ideas lead also to analogous fixed point theorems for *F*-expanding mappings on *G*-metric spaces.

**Definition 3.3**

A mapping $T:X\to X$ from a *G*-metric space (X,G) into itself is said to be

1. *(a)* *F*-expanding of type I on a *G*-metric space *X* if there exist $F\in \mathcal{F}$ and t > 0 such that for all $x, y \in X$,

$$G(x,y,y)>0 \Rightarrow F(G(Tx,Ty,Ty)) \ge F(G(x,y,y)) + t ; \quad (6)$$

*(b)* *F*-expanding of type II on a *G*-metric space *X* if there exist $F \in \mathcal{F}$ and t > 0 such that for all $x,y,z \in X$,

$$G(x,y,z)>0 \Rightarrow F(G(Tx,Ty,Tz)) \ge F(G(x,y,z)) + t.. \quad (7)$$

**Theorem 3.4**

*Let* (X,G) *be a complete G-metric space and* $T:X\to X$ *be a surjective and F-expanding mapping of type I*(*or type II*). *Then T has a unique fixed point*.

**Proof**

Let *T* be an *F*-expanding mapping of type I. From Lemma 1.2, there exists a mapping $T^*: X\to X$ such that $ToT^*$ is the identity mapping on *X*. Let $x,y\in X$ be arbitrary points such that $x\neq y$ and let $\xi=T^*x$ and $\eta=T^*y$. Obviously, $\xi \neq \eta$ , and $G(\xi,\eta,\eta) > 0$. By using (6) applied to $\xi$ and $\eta$, we have

$$F(G(T\xi,T\eta,T\eta)) \ge F(G(\xi,\eta,\eta)) + t.$$

Since $T\xi=T(T^*x) = x$ and $T\eta=T(T^*y) = y$, then

$$F(G(x,y,y)) \ge F(G(T^*x,T^*y,T^*y)) + t,,$$

so $T^*$ is an *F*-contraction of type I on a *G*-metric space (X,G). Theorem 3.2 guarantees that $T^*$ has a unique fixed point $u \in X$. The point *u* is also a fixed point of *T* because $Tu=T(T^*u)=u$.

Now, we prove the uniqueness of the fixed point. Assume that *v* is another fixed point of *T* different from *u*: $Tu = u\neq v = Tv$. This means $G(u,v,v) > 0$, so by (6)

$0 < t \leq F(G(Tu,Tv,Tv)) - F(G(u,v,v))=0$,,

which is a contradiction, and hence u=v.

For *F*-expanding mappings of type II, it is necessary to take $z = y$ and apply the proof for *F*-expanding mappings of type I.

As a corollary of Theorem 3.4, taking F1∈ F, see Examples 1.5, we obtain the following.

**Corollary 3.5[2], Corollary 9.1.4** *Let* $(X,G)$ *be a complete G-metric space and* $T:X\to X$ *be surjective, and let there exist* $\lambda > 1$ *such that*

$G(Tx,Ty,Ty) \geq \lambda G(x,y,y)$ for all $x, y \in X$,

*or*

$G(Tx,Ty,Tz) \geq \lambda G(x,y,z)$ for all $x ,y, z \in X$.

*Then T has a unique fixed point*.

**Remark 3.6** If *T* is not surjective, the previous results are false.Consider $X=(-\infty,-1]\cup[1,\infty)$ endowed with the *G* metric $G(x,y,z)= |x-y| +| x-z| + |y-z|$ for all $x, y ,z \in X$ and the mapping $T: X\to X$ defined by $Tx= -2x$. Then $G(Tx, Ty, Tz) \geq 2G(x,y,z)$ for all $x, y , z \in X$ and *T* has no fixed point.

Now, we will improve some results contained in the book [2]. We will use the following observation: if $T:X\to X$ is a subjective mapping, based on each x0∈ X, there exists a sequence $\{x_n\}$ such that $Tx_{n+1}=x_n$ for all $n \geq 0$

Generally, a sequence {xn} verifying the above condition is not necessarily unique.

**Theorem 3.7**

*Let* $(X,G)$ *be a complete G-metric space, and let* $T:X\to X$ *be a surjective mapping. Suppose that there exist* F∈F *and* $t > 0$ *such that for all* $x,y \in X$**,**

$G(x,Tx,y)>0 \Rightarrow F(G(Tx,T2x,Ty)) \geq F(G(x,Tx,y)) + t$. (8)

*Then T has a unique fixed point*.

**Proof**

Let x0 ∈ X be arbitrary. Since *T* is surjective, there exists x1∈ X such that Tx1 = x0 . By continuing this process, we can find a sequence $\{x_n=Tx_{n+1}\}$ for all n=0,1,2,……. If there exists n0 ∈N U{0} such that $x_{n0}=x_{n0+1}$ , then xn0+1 is a fixed point of *T*.

Now assume that xn≠xn+1 for all $n \geq 1$. Then $G(x_{n+1},x_n,x_n) > 0$ for all $n \geq 1$, and from (8) with $x = x_{n+1}$ and $y=x_n$, we have, for all $n \geq 1$,

$$F(G(x_n,x_{n-1},x_{n-1})) = F(G(Tx_{n+1},T^2x_{n+1},Tx_n))$$
$$\geq F(G(x_{n+1},Tx_{n+1},x_n)) + t = F(G(x_{n+1},x_n,x_n)) + t,$$

and hence

$$t + F(G(x_{n+1},x_n,x_n)) \leq F(G(x_n, x_{n-1}, x_{n-1})). \quad (9).$$

Using (9), the following holds for every $n \geq 1$:

$$F(G(x_{n+1},x_n,x_n)) \leq F(G(x_n,x_{n-1},x_{n-1})) - t$$
$$\leq F(G(x_{n-1},x_{n-2},x_{n-2})) - 2t \leq \text{----} \leq F(G(x_1,x_0,x_0)) - nt. \quad (10)$$

From (10) we obtain

 ISBN: 978-93-86171-46-7

$\lim_{n\to\infty} F(G(xn+1, xn, xn)) = -\infty,$

which together with (*F*2) gives

$$\lim_{n\to\infty} F(G(xn+1, xn, xn)) = 0, \tag{11}$$

From (*F*3) there exists k∈ (0,1) such that

$$\lim_{n\to\infty} F(G(xn+1,xn,xn))^k\, F(G(xn+1, xn, xn)) = 0 \tag{12}$$

By (10), the following holds for all $n \geq 1$:

$[G(x_{n+1},x_n,x_n)]^k F(G(x_{n+1},x_n,x_n)) - [G(x_{n+1},x_n,x_n)]^k F(G(x_1,x_0,x_0))$

$\leq [G(x_{n+1},x_n,x_n)]^k (F(G(x_1,x_0,x_0)) - nt)$

$$- [G(x_{n+1},x_n,x_n)]^k F(G(x1,x0,x0)) = - [G(x_{n+1},x_n,x_n)]^k . nt \tag{13}$$

Letting $n\to\infty$ in (13) and using (11), (12), we obtain

$$\lim_{n\to\infty} [G(x_{n+1},x_n,x_n)]^k .n=0 \quad . (14)$$

Now, let us observe that from (14) there exists n1 $\geq$ 1 such that

$$[G(x_{n+1},x_n,x_n)]^k .n \leq 1 \text{ for all } n \geq n_1.$$

Consequently, we have

$$G(x_{n+1},x_n,x_n) \leq \frac{1}{n1/k} \quad \text{all } n \geq n_1..$$

Since the series $\sum_{i=0}^{\infty} \frac{1}{n1/k}$ converges, for any $\varepsilon > 0$, there exists n2 $\geq$ 1 such that

$\sum_{i=0}^{\infty} \frac{1}{n1/k} < \varepsilon$ In order to show that {xn} is a Cauchy sequence, we consider m > n > max{n1,n2}. From [2], Lemma 3.1.2(4), we get

$G(x_{n+1},x_n,x_n) \leq \sum_{j=n}^{m-1} G(x\,j+1, xj, xj) \leq \sum_{j=n}^{\infty} G(x\,j+1, xj, xj)$

$\leq \sum_{j=n}^{\infty} \frac{1}{j1/k} \leq \sum_{j=n2}^{\infty} \frac{1}{j1/k} \leq \varepsilon.$

Therefore by [2], Lemma 3.2.2 and axiom (G4) {xn} is a Cauchy in a *G*-metric space (X,G).From the completeness of (X,G), there exists u ∈ X such that {xn}→u As *T* is surjective, there exists w ∈ X such that u=Tw. From (8) with $x = x_{n+1}$ and y=w, we have, for all $n \leq 1$,

$F(G(x_{n+1},x_n,u) = F(G(Tx_{n+1},T^2x_{n+1},T_w))$

$\geq F(G(x_{n+1},Tx_{n+1},w)) + t = F(G(x_{n+1},x_n,w)) + t,$

and hence

$$F(G(x_{n+1},x_n,u) > F(G(x_{n+1},x_n,w)) \tag{15}$$

By (*F*1) from (15), we have

$$G(x_{n+1},x_n,u) > G(x_{n+1},x_n,w) \quad \text{for all } n \geq 1 \tag{16}$$

Using the fact that the function *G* is continuous on each variable ([2], Theorem 3.2.2), taking the limit as $n\to\infty$ in the above inequality, we get

$G(u,u,w) = F(G(x_n,x_{n+1},u)) = 0,,$

that is, u= w. Then *u* is a fixed point of *T* because $u = T_w = T_u$.

To prove uniqueness, suppose that u,v ∈ X are two fixed points. If Tu=u≠v =Tv, then G(u,u,v) > 0 So, by (8),

$F(G(u,u,v)) = F(G(Tu,T^2u,Tv))$

$\geq F(G(u,Tu,v)) + t = F(G(u,u,v)) + t,$

which is a contradiction, because t > 0. Hence, u=v .

Taking F1∈ F, see Example 1.5, we obtain the following.

***Corollary 3.8* [2], Theorem 9.1.2** *Let* (X,G)(X,G) *be a complete G-metric space and* T:X→X *be a surjective mapping*. *Suppose that there exists* λ>1 *such that*

$G(Tx,T^2x,Ty) \geq \lambda G(x,Tx,y)$f or allx ,y ∈ X..

*Then T has a unique fixed point*.

Next result does not guarantee the uniqueness of the fixed point.

## Theorem 3.9

*Let* (X,G)(X,G) *be a complete G-metric space, and let* T:X→X *be a surjective mapping*. *Suppose that there exist*F ∈ f *and* t > 0 *such that for all* x,y ∈ X ,

$$G(x,Tx,T^2x) > 0 \Rightarrow F(G(Tx,Ty,T^2y)) \geq F(G(x,Tx,T^2x)) + t. \quad (17)$$

*Then T has a fixed point*.

**<u>Proof</u>**

Let x0 ∈X be arbitrary. Since *T* is surjective, there exists x1∈X such that x0=Tx1. By continuing this process, we can find a sequence $\{xn=Tx_{n+1}\}$ for all n ≥ 0. If there exists n0 ≥ 0 such that $xn_0=xn_0+1$, then xn0+1 is a fixed point of *T*.

Now, assume that xn≠xn+1 for all n ≥ 0 . From (17) with $x=x_{n+1}$ and y=xn,

we have $G(xn+1,Tx_{n+1},T^2x_{n+1}) = G(xn+1,xn,xn-1) > 0$ and

$F(G(x_n,x_{n-1},x_{n-2}))=F(G(Tx_{n+1},Tx_n,T^2x_n))$

$$\geq F(G(x_{n+1},Tx_{n+1},T^2x_{n+1})) + t = F(G(x_{n+1},x_n,x_{n-1})) + t,$$

and hence

$$F(G(x_{n+1},x_n,x_{n-1})) \leq F(G(x_n,x_{n-1},x_{n-2})) - t$$
$$\leq F(G(x_{n-1},x_{n-2},x_{n-3})) - 2t$$
$$\leq F(G(x_2,x_1,x_0)) - (n-1)t. \quad (18)$$

From (18), we obtain

$$\lim_{n\to\infty} F\big(G(xn+1,xn,xn-1)\big) = -\infty,$$

which together with (*F*2) gives

$$\lim_{n\to\infty} F\big(G(xn+1,xn,xn-1)\big) = 0,$$

Mimicking the proof of Theorem 3.7, we obtain

$$\lim_{n\to\infty} F\big(G(xn+1,xn,xn-1)\big)]k.(n-1) = 0;$$

and consequently, there exists $n_1 \geq 1$ such that

$$G(x_{n+1},x_n,x_{n-1}) \leq \frac{1}{(n-1)1/k} \text{ for all } n > n_1$$

ISBN: 978-93-86171-46-7

Since the series $\sum_{i=1}^{\infty} \frac{1}{i1/k}$ converges, for any ε > 0, there exists n2 ≥ 1 such that $\sum_{i=n2}^{\infty} \frac{1}{i1/k} < \varepsilon$ . In order to show that {xn} is a Cauchy sequence, we consider m>n>max{n1,n2}. From [2], Lemma 3.1.2(4) and axioms (G3), (G4), we get

$$G(x_m,x_n,x_n) \leq \sum_{j=n}^{m-1} G(x\,j+1, xj, xj) \leq \sum_{j=n}^{\infty} G(x\,j+1, xj, xj)$$

$$\leq \sum_{j=n}^{\infty} G(x\,j+1, xj, xj-1) \leq \sum_{j=n}^{\infty} \frac{1}{j1/k} \leq \sum_{j=n2}^{\infty} \frac{1}{j1/k} < \varepsilon$$

Therefore, by [2], Lemma 3.2.2, {xn}{xn} is a Cauchy in a *G*-metric space (X,G). From the completeness of (X,G), there exists u ∈ X such that {xn}→u. As *T* is surjective, there exists w ∈ X such that u=Tw. From (17) with x=w and y=xn+1, we have

$$F(G(u,x_n,x_{n-1})) = F(G(Tw,Tx_{n+1},T^2x_{n+1})) \geq F(G(w,Tw,T^2w)) + t ,$$

so

$$F(G(w,Tw,T^2w)) \leq F(G(u,x_n,x_{n-1})) - t < F(G(u,x_n,x_{n-1}))$$

Using (*F*1), we have

$$G(w,Tw,T^2w) < G(u,x_n,x_{n-1}) \text{for all } n \geq 1.$$

Using the fact that the function *G* is continuous on each variable ([2], Theorem 3.2.2), taking the limit as n→∞n→∞ in the above inequality, we get

$$G(w,Tw,T^2w)= \lim_{n\to\infty} G(u,x_n,x_{n-1}) = 0$$

that is, $w=Tw=T^2w$. Hence, u=Tu.

Taking $F_1$∈ F, see Examples 1.5, we obtain the following**.**

## IV CONCLUSION

**Theorme:-** *Let* (X,G)(X,G) *be a complete G-metric space and* T:X→X *be a surjective mapping*. *Suppose that there exists* λ > 1 *such that*

$G(Tx,Ty,T^2y) \geq \lambda G(x,Tx,T^2x)$ for all x, y ∈ X..

*Then T has*, *a fixed point*.with F –Expanding Mapping .

## REFERENCES


1. Wardowski, D: Fixed points of a new type of contractive mappings in complete metric spaces. Fixed Point Theory Appl.**2012**, Article ID 94 (2012)
2. Agarwal, RP, Karapinar, E, O'Regan, D, Roldán-López-de-Hierro, AF: Fixed Point Theory in Metric Type Spaces. Springer, Switzerland (2015)
3. Samet, B, Vetro, C, Vetro, F: Remarks on *G*-metric spaces. Int. J. Anal. **2013**, Article ID 917158 (2013)
4. R. P. AGARWAL and D. O'REGAN, Difference equations in abstract spaces, J. Austral. Math. Soc. Ser. A, 64 (1998), 277–284.
5. J. BANAS and K. GOEBEL, Measures of noncompactness in Banach spaces, Marcel Dekker, 1980

ISBN: 978-93-86171-46-7

6. Hussain, N, Al-Mezel, S, Salimi, P: Fixed points for ψ-graphic contractions with application to integral equations. Abstr. Appl. Anal. **2013**, Article ID 575869 (2013)
7. Dhage, BC: A fixed point theorem for multi-valued mappings in ordered Banach spaces with applications I. Nonlinear Anal. Forum **10**(1), 105-126 (2005)
8. Yogesh Kumar : continuation methods for contractive and non expansive mapping (function),j. Ijarse-09,sept2016 ,2319-8354