International Conference on Recent Innovations in Engineering, Applied Sciences and Management

(IETE) Institution of Electronics and Telecommunication Engineers, New Delhi, India

(EAM-17)

10th September 2017, www.conferenceworld.in

ISBN: 978-93-86171-64-1

ON DISCONNECTED SETS VIA IDEALS

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ABSRACT

*We will introduce disconnected sets with respect to an ideal in topological space and give its relationship with disconnected sets with respect to given topology and its *-topology. We also use these sets to obtain various properties of $\mathfrak{I}$-compact sets.*

Keywords and phrases*: $\mathfrak{I}$-disconnected, $\mathfrak{I}$-compact, S_2 mod $\mathfrak{I}$, $\mathfrak{I}$-regular, $\mathfrak{I}$-normal.*

2000 **MSC**: 54D10, 54D15.

I. INTRODUCTION

In [1], Csàszàr, introduced disconnected sets as well as S_1 and S_2 spaces and discussed some properties of compact sets using these sets. On the other hand separation and covering axioms with respect to an ideal and various properties and characterizations were also discussed by many authors, for instance see ([2],[3],[6]). Ideals in topological spaces were introduced by Kuratowski[5] and further studied by Vaidyanathaswamy[7]. Corresponding to an ideal a new topology $\tau^*(\mathfrak{I}, \tau)$ called the *-topology was given which is generally finer than the original topology having the kuratowski closure operator $cl^*(A) = A \cup A^*(\mathfrak{I},\tau)$[8], where $A^*(\mathfrak{I},\tau) = \{x \in X : U \cap A \notin \mathfrak{I}$ for every open subset U of x in X called a local function of A with respect to $\mathfrak{I}$ and τ. We will write τ^* for $\tau^*(\mathfrak{I}, \tau)$.

The following section contains some definitions and results that will be used in our further sections.

Definition 1.1.[5]: Let (X, τ) be a topological space. An ideal $\mathfrak{I}$ on X is a collection of non-empty subsets of X such that (a) $\phi \in \mathfrak{I}$ (b) $A \in \mathfrak{I}$ and $B \in \mathfrak{I}$ implies $A \cup B \in \mathfrak{I}$ (c) $B \in \mathfrak{I}$ and $A \subset B$ implies $A \in \mathfrak{I}$.

Definition 1.2.[1]: A topological space (X, τ) is said to be S_2 space if for every pair of distinct points x and y, whenever one of them has a open set not containing the other then there exist disjoint open subsets containing them.

Definition 1.3.[1]: Let (X,τ) be a topological space. Any two subsets A and B are said to be disconnected if there exist disjoint open subsets of X containing them.

Definition 1.4.[2]: An ideal space $(X,\tau,\mathfrak{I})$ is said to be compactness modulo an ideal or simply $\mathfrak{I}$-compact if for every open cover $\{V_\alpha | \alpha \in \Delta\}$ of X, there is a finite subset Δ_0 of Δ such that $X - \cup\{V_\alpha | \alpha \in \Delta_0\} \in \mathfrak{I}$.

Definition 1.5.[3]: An ideal space $(X,\tau,\mathfrak{I})$ is said to be $\mathfrak{I}$-regular if for any closed subset F of X and any point $x \in X$ whenever $x \notin F$ then there exist disjoint open subsets U and V such that $x \in U$ and $F-V \in \mathfrak{I}$.

Definition 1.6.[6] : An ideal space $(X,\tau,\mathfrak{I})$ is said to be $\mathfrak{I}$-normal if for any disjoint closed subsets F and F' of X there exist disjoint open subsets U and V such that $F-U \in \mathfrak{I}$ and $F'-V \in \mathfrak{I}$.

Theorem 1.7.[2]: Let $(X,\tau,\mathfrak{I})$ be $\mathfrak{I}$-compact space and A be τ^*-closed subset of X. Then A is $\mathfrak{I}$-compact.

II. RESULTS

We begin by introducing the following definitions of disconnected sets with respect to an ideal.

Definition 2.1.: Let $(X,\tau,\mathfrak{I})$ be an ideal space and A,B be two subsets of X. Then A and B are said to be $\mathfrak{I}$-disconnected if there exist open subsets U and V such that $A-U \in \mathfrak{I}$, $B-V \in \mathfrak{I}$ and $U \cap V \in \mathfrak{I}$.

Definition 2.2.: Let $(X,\tau,\mathfrak{I})$ be an ideal space and A be any subset of X. Then any point $x \in X$, x and A are said to be $\mathfrak{I}$-disconnected if there exist open subsets U and V such that $x \in U$, $A-V \in \mathfrak{I}$ and $U \cap V \in \mathfrak{I}$.

Remark 2.3.: It can be easily seen that if A and B are disconnected sets then A and B are also $\mathfrak{I}$-disconnected, since $\phi \in \mathfrak{I}$. But the converse is not true as can be seen from the example below:

Example 2.4.: Let $X=\{a,b,c\}$, $\tau=\{\phi, \{a\}, \{b\}, \{a,b\}, X\}$ and $\mathfrak{I}=\{\phi,\{c\}\}$. Consider $A=\{a,c\}$ and $B=\{b,c\}$. Then A and B are not disconnected because X is the only open set containing A and B. But there exist open sets $U=\{a\}$ and $V=\{b\}$ such that $A-U \in \mathfrak{I}$, $B-V \in \mathfrak{I}$ and $U \cap V \in \mathfrak{I}$.

The above Example 2.4 also shows that A and B are $\mathfrak{I}$-disconnected but A and B are not disconnected with respect to *-topology since $\tau=\tau^*$. And the following Example 2.5 shows that the converse is not true.

Example 2.5.: Let $X=\{a,b,c\}$, $\tau=\{\phi, \{a\}, \{b\}, \{a,b\}, X\}$ and $\mathfrak{I}=\{\phi,\{a\},\{b\},\{a,b\}\}$ and so $\tau^*=\wp(X)$. Consider $A=\{a,c\}$ and $B=\{b,c\}$. Then A and B are obviously disconnected with respect to τ^*, but A and B are not $\mathfrak{I}$-disconnected.

Remark 2.6.: It can be easily seen that if A and B are disconnected with respect to τ then A and B are also disconnected with respect to τ^*. But the above Example 2.5 shows that the converse is not true.

Therefore, for any two subsets A and B in an ideal topological space the relationship of disconnected sets with respect to topological spaces can be seen below.

a) A and B are disconnected in $(X, \tau) \Rightarrow$ A and B are $\mathfrak{I}$-disconnected.

b) A and B are $\mathfrak{I}$-disconnected $\not\Rightarrow$ A and B are disconnected in (X, τ).

c) A and B are disconnected in $(X, \tau) \Rightarrow$ A and B are disconnected in $(X, \tau^*) \not\Rightarrow$ A and B are disconnected in (X, τ).

d) A and B are $\mathfrak{I}$-disconnected $\not\Leftrightarrow$ A and B are disconnected in (X, τ^*).

iii. Applications of $\mathfrak{I}$-disconnected sets

Theorem 3.1.: Let $(X,\tau,\mathfrak{I})$ be an ideal space and A be any subset of X and K is $\mathfrak{I}$-compact subset of X such that A and every point of K are $\mathfrak{I}$-disconnected then A and K are $\mathfrak{I}$-disconnected.

Proof: Let $y \in K$ be any element, then A and every point of K are $\mathfrak{I}$-disconnected implies that there exist open subsets U_y and V_y such that $y \in U_y$, $A- V_y \in \mathfrak{I}$ and $U_y \cap V_y \in \mathfrak{I}$. Therefore, $K \subset \bigcup\{U_y : y \in K\}$. But K is $\mathfrak{I}$-compact implies that there exist finite subset K' of K such that $K-\bigcup\{U_y : y \in K'\} \in \mathfrak{I}$. Now consider $U= \bigcup\{U_y : y \in K'\}$ and $V = \bigcap\{V_y : y \in K'\}$ then $K-U \in \mathfrak{I}$ and $U \cap V \in \mathfrak{I}$. Also $A- V_y \in \mathfrak{I}$ for all $y \in K'$ implies that $\bigcup\{A- V_y : y \in K'\} \in \mathfrak{I}$ and so $A-\bigcap\{V_y : y \in K'\} \in \mathfrak{I}$ and so $A-V \in \mathfrak{I}$. Therefore, there exist open subsets U and V such that $A-V \in \mathfrak{I}$, $K-U \in \mathfrak{I}$ and $U \cap V \in \mathfrak{I}$ and hence A and K are $\mathfrak{I}$-disconnected.

Corollary 3.2.: Let $(X,\tau,\mathfrak{I})$ be an ideal space and A be any subset of X and K is $\mathfrak{I}$-compact subset of X such that for the subset A and every point x of K, there exist disjoint open subsets U_x and V_x such that $x \in U_x$ and $A-V_x \in \mathfrak{I}$ then there exist disjoint open subsets G and H such that $A-G \in \mathfrak{I}$ and $K-H \in \mathfrak{I}$.

Proof: Proof is similar to that of Theorem 3.1 and hence is omitted.

For our next results firstly we define S_2 mod $\mathfrak{I}$ spaces.

Definition 3.3. : An ideal space $(X,\tau,\mathfrak{I})$ is said to be **S_2 mod $\mathfrak{I}$** if for every pair of distinct points a and b in X, whenever one of them has open subset not containing the other then there exist open subsets G and H containing a and b respectively such that $G \cap H \in \mathfrak{I}$.

Theorem 3.4.: Let $(X,\tau,\mathfrak{I})$ be S_2 mod $\mathfrak{I}$ space and K be $\mathfrak{I}$-compact closed subset of X. Then for any point $x \notin K$, x and K are $\mathfrak{I}$-disconnected.

Proof: Let K be $\mathfrak{I}$-compact subset of X and $x \notin K$ be any element and so X-K is open set containing x but not containing the elements of K, since K is closed. Further, X is S_2 mod $\mathfrak{I}$ space implies that for all $y \in K$, there exist open subsets U_y and V_y containing x and y respectively such that $U_y \cap V_y \in \mathfrak{I}$. Therefore, for the subset A={x}, A and any point of K are $\mathfrak{I}$-disconnected and so by Theorem 3.1, A and K are $\mathfrak{I}$-disconnected i.e. x and K are $\mathfrak{I}$-disconnected.

Corollary 3.5.: Let $(X,\tau, \mathfrak{I})$ be S_2 space and K be $\mathfrak{I}$-compact closed subset of X. Then for any point $x \notin K$ there exist disjoint open subsets U and V such that $x \in U$ and $K\text{-}V \in \mathfrak{I}$.

Proof: Proof is similar to that of Theorem 3.4 and follows from the fact that in S_2 space for any two distinct points, if one of them has open set not containing the other then there exist disjoint open subsets containing them and hence is omitted.

Theorem 3.6.: Let $(X,\tau, \mathfrak{I})$ be an ideal space and (X,τ^*) be S_2. If K is an $\mathfrak{I}$-compact τ^*-closed subset of X, then for any point $x \notin K$, x and K are $\mathfrak{I}$-disconnected.

Proof: Let K be $\mathfrak{I}$-compact subset of X and $x \notin K$ be any element and so X-K is τ^*-open set containing x but not containing the elements of K, since K is τ^*-closed. Further, (X, τ^*) is S_2 space implies that for all $y \in K$, there exist disjoint τ^*-open subsets U_y and V_y containing x and y respectively. But the collection $\beta = \{ V\text{-}I : V \in \tau$ and $I \in \mathfrak{I}\}$ will form a basis for the *-topology τ^* [4]. Therefore, there exist open subsets G_y and H_y and $I_y, I'_y \in I$ such that $x \in G_y\text{-} I_y \subset U_y$ and $y \in H_y - I'_y \subset V_y$ and so $(G_y \cap H_y)\text{-}(I_y \cup I'_y) = (G_y - I_y)\cap(H_y - I'_y) \subset U_y \cap V_y = \phi$ implies that $G_y \cap H_y \in \mathfrak{I}$. Hence for the subset A={x}, A and any point of K are $\mathfrak{I}$-disconnected and so by Theorem 3.1, A and K are $\mathfrak{I}$-disconnected i.e. x and K are $\mathfrak{I}$-disconnected.

Theorem 3.7.: Let $(X,\tau, \mathfrak{I})$ be S_2 mod $\mathfrak{I}$ space and K_1, K_2 are $\mathfrak{I}$-compact subsets of X. If K_1 is closed and $K_1 \cap K_2=\phi$, then K_1 and K_2 are $\mathfrak{I}$-disconnected.

Proof: Let K_1, K_2 be two $\mathfrak{I}$-compact subsets of X and also K_1 be closed subset of X such that $K_1 \cap K_2=\phi$ then for all $x \in K_2$, $x \notin K_1$. Therefore, by Theorem 3.3, we have x and K_1 are $\mathfrak{I}$-disconnected. Further, by Theorem 3.1, K_1 and every point of $\mathfrak{I}$-compact subset K_2 are disconnected implies that K_1 and K_2 are $\mathfrak{I}$-disconnected.

Corollary 3.8.: Let $(X,\tau, \mathfrak{I})$ be S_2 space and K_1, K_2 are $\mathfrak{I}$-compact subsets of X. If K_1 is closed and $K_1 \cap K_2=\phi$, then there exist disjoint open subsets G and H such that $K_1\text{-}G \in \mathfrak{I}$ and $K_2\text{-}H \in \mathfrak{I}$.

Corollary 3.9.: Let $(X,\tau, \mathfrak{I})$ be an ideal space and (X,τ^*) be S_2. If K_1, K_2 are $\mathfrak{I}$-compact subsets of X and K_1 is τ^*-closed such that $K_1 \cap K_2=\phi$, then K_1 and K_2 are $\mathfrak{I}$-disconnected.

Theorem 3.10.: Let $(X,\tau, \mathfrak{I})$ be S_2 mod $\mathfrak{I}$ space and K be an $\mathfrak{I}$-compact and F be closed subset of X such that $K \cap F=\phi$ then $cl^*(K) \cap F=\phi$.

Proof: Let K be an $\mathfrak{I}$-compact subset of X and F be closed subset of X such that $K \cap F=\phi$. If one of K and F is empty then nothing to prove. So consider the case when both K and F are non-empty. Let $y \in F$ be any element then we have to prove $y \notin cl^*(K)$. Now $K \cap F=\phi$ implies that $K \subset X\text{-}F$ and so X-F is open set containing the elements of K but not containing y. Further, X is S_2 mod $\mathfrak{I}$ implies that for all $x \in K$, there exist open subsets U_x

and V_x containing x and y respectively such that $U_x \cap V_x \in \mathfrak{T}$. This implies that for the subset A={y} and any point of $\mathfrak{T}$-compact set K are $\mathfrak{T}$-disconnected and so by Theorem 3.1, A and K are $\mathfrak{T}$-disconnected. Therefore, there exist open subsets G and H such that $y \in G, K-H \in \mathfrak{T}$ and $G \cap H \in \mathfrak{T}$ and so $G \cap K \in \mathfrak{T}$. Thus $y \notin K^*$. Also $y \notin K$ implies that $cl^*(K) \cap F = \phi$.

Theorem 3.11.: Let $(X,\tau,\mathfrak{T})$ be $\mathfrak{T}$-regular space and K be $\mathfrak{T}$-compact and F be closed subset of X such that $K \cap F = \phi$. Then K and F are $\mathfrak{T}$-disconnected.

Proof: Let K be $\mathfrak{T}$-compact and F be closed subset of X such $K \cap F = \phi$. Then for all $x \in K$, $K \cap F = \phi$ implies that $x \notin F$ and so X is $\mathfrak{T}$-regular and F is closed implies that there exist disjoint open subsets U and V such that $x \in U$ and $F-V \in \mathfrak{T}$. Therefore, F and any point of $\mathfrak{T}$-compact subset K are $\mathfrak{T}$-disconnected and so Theorem 3.1 implies that K and F are $\mathfrak{T}$-disconnected.

Theorem 3.12.: Every $\mathfrak{T}$-compact S_2 space is $\mathfrak{T}$-normal.

Proof: Let $(X,\tau,\mathfrak{T})$ be $\mathfrak{T}$-compact and S_2 space. Consider A and B be any two disjoint closed subsets of X, then A and B are also $\mathfrak{T}$-compact using Theorem 1.7. Therefore, Corollary 3.8 implies that there exist disjoint open subsets G and H such that $A-G \in \mathfrak{T}$ and $B-H \in \mathfrak{T}$ and hence X is $\mathfrak{T}$-normal.

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