COMMON RANDOM FIXED POINTS RESULTS FOR(ψ,φ)-CONTRACTIONS VIATHE CONCEPT OF $\mathcal{C}$-CLASS FUNCTIONS

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Abstract:*We prove a new approach for some common random fixed point results in partially ordered complete separable metric spaces for weakly increasing self-mappings satisfying* (ψ,φ)*-contractions via the concept of* $\mathcal{C}$*-class functions.*

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1. Introduction

The existence of fixed points for self-mappings in partially ordered sets has been consideredin [1,2], where some applications to matrix equations are presented. This resultwas extended byNieto et al.[3] and Nieto and Rodriguez-Lopez [4, 5] in partially orderedsets and applied to study ordinary differential equations.

The problem of fixed points for random mappings was initiated by the Prague school of probability research. The first results were studied in 1955-1956by Špaček and Hanš in the context of Fredholm integral equations with random kernel. In a separable metric space, random fixed point theorems for contraction mappings were proved by Hanš [8, 9], Hanš and Špaček [10] and Mukherjee [11, 12]. Then random fixed point theorems of Schauder or Krasnosel'skii type were given by Mukherjea (*cf.* Bharucha-Reid [6], p. 110), Bharucha-Reid [13] and Itoh [14]. Now it has become a full-fledged research area and a vast amount of mathematical activities have been carried out in this direction (see, for examples, [15–18]). The existence of a random fixed point for mappings in partially ordered metric spaces and partially ordered probabilistic metric spaces was studied, for example, in [19, 20].In 2014, Ansari [1] introduced the concept of C-classfunctions and proved the unique fixed point theorems for certain contractive mappings with respect to the$\mathcal{C}$-class functions.

The goal of this paper is to establish a common random fixed point results in partially ordered complete separable metric spaces for weakly increasing self mappings satisfying (ψ, φ)-contractions via the concept of C-class functions. Some corollaries are also presented for particular cases of the C-function.

2. Mathematical Preliminaries

The triple$(X, d, \preceq)$is called a partially ordered metric space if $(X, \preceq)$is a partially ordered set and(X, d) is a metric space. Further, if(X, d) is complete metric space, and then the triple$(X, d, \preceq)$ is called a partially ordered complete metric space.

Definition 2.1 Let (X, d) be a metric space endowed witha partial order $\preceq$. Let $\{x_n\}$and $z$be in X. $(X, d, \preceq)$is said tobe regular if $x_n \to z$ and $\{x_n\}$is non-decreasing; then $x_n \preceq z$for all $n \in \mathbb{N}$.

Let (X, β_X) be a separable Banach space, where $\beta_X$is a σ-algebra of Borel subsets of X, and let (Ω, β, μ) denote a complete probability measure space with measure $\mu$and βbe a σ-algebra of subsets of Ω.

Definition 2.2A measurable mapping $\xi: \Omega \to X$is said to be an X-valued random variable if the inverse image under the mapping $x$of every Borel set $B$of Xbelongs to β, that is, $\xi^{-1}(B) \in \beta$for all $B \in \beta_X$.

Definition 2.3A measurable mapping $\xi: \Omega \to X$ is said to be a finitely-valued random variable if it is constant on each finite number of disjoint sets $A_i \in \beta$and is equal to 0 on $\Omega - (\cup_{i=1}^{n} A)$. $\xi$is called a simple random variable if it is finitely valued and $\mu\{\omega: \|\xi(\omega)\| > 0\} < \infty$.

Definition 2.4A measurable mapping $\xi: \Omega \to X$is said to be a strong random variable if there exists a sequence $\{\xi_n(\omega)\}$of simple random variables which converges to $\xi(\omega)$ almost surely, that is, there exists a set $A_0 \in \beta$with $\mu(A_0) = 0$ such that

$\lim_{n\to\infty} \xi_n(\omega) = \xi(\omega), \omega \in \Omega - A_0$.

Definition 2.5A measurable mapping $\xi: \Omega \to X$is said to be a weak random variable if the function $\xi^*(\xi(\omega))$ is a real-valued random variable for each $\xi^* \in X^*$, the space X^* denoting the first normed dual space of X.

Definition 2.6 Let Y be another Banach space. A measurable mapping $f: \Omega \times X \to Y$is said to be a random mapping if $f(\omega, \xi) = Y(\omega)$ is a Y-valued random variable for every $\xi \in X$.

Definition 2.7A measurable mapping $f: \Omega \times X \to Y$is said to be a continuous random mapping if the set of all $\omega \in \Omega$for which $f(\omega, \xi)$ is a continuous function of ξhas measure one.

Definition 2.8A mapping measurable$f: \Omega \times X \to Y$is said to be demi-continuous at the$\xi \in X$ if

$\|\xi_n - \xi\| \to 0$ implies $f(\omega, \xi_n) \xrightarrow{weakly} f(\omega, \xi)$

almost surely.

Definition 2.9 An equation of the type $f(\omega, \xi(\omega)) = \xi(\omega)$, where $f: \Omega \times X \to X$is a random mapping, is called a random fixed point equation.

Definition 2.10 Anymeasurable mapping $\xi: \Omega \to X$which satisfies the random fixed point equation $f(\omega, \xi(\omega)) = \xi(\omega)$ almost surely is said to be a wide sense solution of the fixed point equation.

Definition 2.11 Any X-valued random variable $\xi(\omega)$ which satisfies

$\mu\{\omega : f(\omega, \xi(\omega)) = \xi(\omega)\} = 1$

is said to be a random solution of the fixed point equation or a random fixed point of f.

Definition 2.12A measurable mapping $\xi: \Omega \to X$ is called a random fixed point of a random operator $f: \Omega \times X \to X$ if $\xi(\omega) = f\big(\omega, \xi(\omega)\big)$ for every $\omega \in \Omega$.

Definition 2.13A measurable mapping $\xi: \Omega \to X$ is called a random coincidence of random operators $T, f: \Omega \times X \to X$ if

$T\big(\omega, \xi(\omega)\big) = f\big(\omega, \xi(\omega)\big)$ for every $\omega \in \Omega$.

Definition 2.14A measurable mapping $\xi: \Omega \to X$ is called a random common fixed point of random operators $f, g : \Omega \times X \to X$ if

$T\big(\omega, \xi(\omega)\big) = f\big(\omega, \xi(\omega)\big) = \xi(\omega)$ for every $\omega \in \Omega$.

Example 2.15 Let Xbe the set of all real numbers and let Ebe a non-measurable subset of X. Let $f: \Omega \times X \to Y$be a random mapping defined as

$f(\omega, \xi(\omega),) = \xi^2(\omega) + \xi(\omega) - 1$

for all $\omega \in \Omega$. In this case, the real-valued function $\xi(\omega)$, defined as $\xi(\omega) = 1$ for all $\omega \in \Omega$, is a random fixed point of f. However, the real-valued function $y(\omega)$ defined as

$y(\omega) = \begin{cases} -1, & \omega \notin E, \\ 1 & \omega \in E \end{cases}$

is a wide sense solution of the fixed point equation $f(\omega, \xi(\omega)) = \xi(\omega)$without being a random fixed point of f.

Definition 2.16 Let $(X, \preceq, d)$ is a partially ordered separable metric space.

(1) A random operator $f: \Omega \times X \to X$is said to be monotone non-decreasing if for all $x, y \in X$,
$x \preceq y \Rightarrow f\big(\omega, x(\omega)\big) \preceq f\big(\omega, y(\omega)\big), \omega \in \Omega$.

(2) Two random operators $f, g: \Omega \times X \to X$ is said to be weakly increasing if for all $x \in X$and $\omega \in \Omega$,
$f\big(\omega, x(\omega)\big) \preceq g\Big(\omega, f\big(\omega, x(\omega)\big)\Big)$and $g\big(\omega, x(\omega)\big) \preceq f\Big(\omega, g\big(\omega, x(\omega)\big)\Big)$.

Ansari [21] introduced the class of $\mathcal{C}$-functions whichcovers a large class of contractive conditions.

Definition 2.17[21] A mapping $\mathcal{F}: [0, \infty)^2 \to \mathcal{R}$ is called C-class function if it is continuous and satisfies following axioms:

(1) $\mathcal{F}(s, t) \leq s$for all $s, t \in [0, \infty)$;

(2) $\mathcal{F}(s, t) = s$ implies that either $s = 0$ or $t = 0$ for all $s, t \in [0, \infty)$.

Mention that any C-function $\mathcal{F}$verifies $\mathcal{F}(0, 0) = 0$. We denote by $\mathcal{C}$the set of C-class functions.

Example 2.18[21] The following functions $\mathcal{F}: [0, \infty)^2 \to \mathcal{R}$ are elements of $\mathcal{C}$. For all $s, t \in [0, \infty)$, consider

(1) $\mathcal{F}(s, t) = s - t$;

(2) $F(s, t) = ms, 0 < m < 1$;

(3) $\mathcal{F}(s, t) = s/(1 + t)^r$, where $r \in (0, \infty)$;

(4) $\mathcal{F}(s,t) = log_a\left(\frac{t+a^s}{1+t}\right), a > 1;$

(5) $\mathcal{F}(s,t) = log_a\left(\frac{1+a^s}{2}\right);$

(6) $\mathcal{F}(s,t) = (s+l)^{\frac{1}{(1+t)^r}} - l, l > 1, r \in (0,\infty);$

(7) $\mathcal{F}(s,t) = s\log_{t+a} a, a > 1;$

(8) $\mathcal{F}(s,t) = s - \left(\frac{1+s}{2+s}\right)\left(\frac{t}{1+t}\right);$

(9) $\mathcal{F}(s,t) = s\beta(s)$, where $\beta: [0,\infty) \to [0,1)$ is continuous;

(10) $\mathcal{F}(s,t) = s - \frac{t}{k+t};$

(11) $\mathcal{F}(s,t) = s - \varphi(s)$,where $\varphi: [0,\infty) \to [0,\infty)$ is a continuous function such that $\varphi(t) = 0 \Leftrightarrow t = 0;$

(12) $\mathcal{F}(s,t) = sh(s,t)$, where $h: [0,\infty) \times [0,\infty) \to [0,\infty)$ is a continuous function such that $h(t,s) < 1$ for all $t, s > 0;$

(13) $\mathcal{F}(s,t) = s - \left(\frac{2+t}{1+t}\right)t;$

(14) $\mathcal{F}(s,t) = \sqrt[n]{ln(1+s^n)};$

(15) $\mathcal{F}(s,t) = \varphi(s)$, where $\varphi: [0,\infty) \to [0,\infty)$ is a is a upper semi-continuous function such that $\varphi(0) = 0$ and $\varphi(t) < t$ for $t > 0;$

(16) $\mathcal{F}(s,t) = \frac{s}{(1+s)^r}, r \in (0,\infty).$

(17) $F(s,t) = \vartheta(s)$, where $\vartheta: \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$is a generalized Mizoguchi-Takahashi type function;

(18) $F(s,t) = \frac{s}{\Gamma(1/2)}\int_0^\infty \frac{e^{-x}}{\sqrt{x}+t}dx$,where $\Gamma$is the Euler Gamma function.

3. Main Result

First, we introduce an auxiliary lemma as follows.

Lemma 3.1 Let$(\Omega,\Sigma,\mu)$is a complete probability measure space,(X,d) be a separable metric space,and$\{\xi_n(\omega):\omega \in \Omega\}$be asequence ofmeasurable mappings from Ω to X such that $\{d(\xi_n(\omega),\xi_{n+1}(\omega))\}$is decreasing and

$\lim_{n\to\infty} d(\xi_n(\omega),\xi_{n+1}(\omega)) = 0$ (3.1)

If $\{\xi_{2n}(\omega)\}$ is not a Cauchy sequence, then there exist an $\epsilon(\omega) > 0$ and $\{m_i\}$,$\{n_i\}$of positive integers such that the four sequences $\left\{d\left(\xi_{2n_i}(\omega),\xi_{2m_i}(\omega)\right)\right\}, \left\{d\left(\xi_{2n_i-1}(\omega),\xi_{2m_i}(\omega)\right)\right\}, \left\{d\left(\xi_{2n_i}(\omega),\xi_{2m_i-1}(\omega)\right)\right\}$ and $\left\{d\left(\xi_{2n_i-1}(\omega),\xi_{2m_i-1}(\omega)\right)\right\}$ tend to $\epsilon(\omega)$ when $i \to \infty$.

Proof:Assume that$\{\xi_{2n}(\omega):\omega \in \Omega\}$is not a Cauchy sequence, it is sufficient to prove that $\{\xi_{2n}(\omega)\}$ is a Cauchy sequence. So there exist $\epsilon(\omega) > 0$ for which we can find two subsequences of positive integers $\{m_i\}$ and $\{n_i\}$ for positive integer i, we

$m_i > n_i > i,\ d\left(\xi_{2n_i}(\omega),\xi_{2m_i}(\omega)\right) \geq \epsilon(\omega),\ i \geq 1, \omega \in \Omega$ (3.2)

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Further, we can choose m_i to be smallest integer with $m_i > n_i$ for which (3.2) holds. Then

$$d\left(\xi_{2n_i}(\omega), \xi_{2m_i-2}(\omega)\right) < \epsilon(\omega) \qquad (3.3)$$

Using (3.2), (3.3) and the triangle inequality, we obtain

$$\epsilon(\omega) \le d\left(\xi_{2n_i}(\omega), \xi_{2m_i}(\omega)\right)$$

$$\le d\left(\xi_{2n_i}(\omega), \xi_{2m_i-2}(\omega)\right) + d\left(\xi_{2m_i-2}(\omega), \xi_{2m_i-1}(\omega)\right)$$

$$+d\left(\xi_{2m_i-1}(\omega), \xi_{2m_i}(\omega)\right)$$

$$\le \epsilon(\omega) + \delta_{2m_i-2}(\omega) + \delta_{2m_i-1}(\omega) \qquad (3.4)$$

On letting the limit as $i \to \infty$ in the above inequality and using (3.1), we get

$$lim_{i\to\infty}\, d\left(\xi_{2n_i}(\omega), \xi_{2m_i}(\omega)\right) = \epsilon(\omega), \ \ \omega \in \Omega \qquad (3.5)$$

addition, by the triangle inequality, we have

$$d\left(\xi_{2n_i}(\omega), \xi_{2m_i}(\omega)\right) \le d\left(\xi_{2n_i}(\omega), \xi_{2n_i-1}(\omega)\right) + d\left(\xi_{2n_i-1}(\omega), \xi_{2m_i-1}(\omega)\right)$$

$$+ d\left(\xi_{2m_i-1}(\omega), \xi_{2m_i}(\omega)\right)$$

$$= \delta_{2n_i-1}(\omega) + d\left(\xi_{2n_i-1}(\omega), \xi_{2m_i-1}(\omega)\right) + \delta_{2m_i-1}(\omega) \qquad (3.6)$$

$$d\left(\xi_{2n_i-1}(\omega), \xi_{2m_i-1}(\omega)\right) \le d\left(\xi_{2n_i-1}(\omega), \xi_{2n_i}(\omega)\right) + d\left(\xi_{2n_i}(\omega), \xi_{2m_i}(\omega)\right)$$

$$+ d\left(\xi_{2m_i}(\omega), \xi_{2m_i-1}(\omega)\right)$$

$$= \delta_{2n_i-1}(\omega) + d\left(\xi_{2n_i}(\omega), \xi_{2m_i}(\omega)\right) + \delta_{2m_i-1}(\omega) \qquad (3.7)$$

Letting the limit as $i \to \infty$ in the above two inequality, using (3.1) and (3.5), we get

$$lim_{i\to\infty}\, d\left(\xi_{2n_i-1}(\omega), \xi_{2m_i-1}(\omega)\right) = \epsilon(\omega), \ \ \omega \in \Omega \qquad (3.8)$$

Also

$$\left|d\left(\xi_{2n_i-1}(\omega), \xi_{2m_i}(\omega)\right) - d\left(\xi_{2n_i}(\omega), \xi_{2m_i}(\omega)\right)\right| \le d\left(\xi_{2n_i}(\omega), \xi_{2n_i-1}(\omega)\right) \qquad (3.9)$$

$$\left|d\left(\xi_{2n_i}(\omega), \xi_{2m_i-1}(\omega)\right) - d\left(\xi_{2n_i}(\omega), \xi_{2m_i}(\omega)\right)\right| \le d\left(\xi_{2m_i}(\omega), \xi_{2m_i-1}(\omega)\right) \qquad (3.10)$$

On letting the limit as $i \to \infty$ in (3.9) and (3.10), using (3.1) and (3.5), we get

$$\left|lim_{i\to\infty}\, d\left(\xi_{2n_i-1}(\omega), \xi_{2m_i}(\omega)\right) - \epsilon(\omega)\right| \le 0,$$

$$\left|lim_{i\to\infty}\, d\left(\xi_{2n_i}(\omega), \xi_{2m_i-1}(\omega)\right) - \epsilon(\omega)\right| \le 0.$$

Hence

$$lim_{i\to\infty}\, d\left(\xi_{2n_i-1}(\omega), \xi_{2m_i}(\omega)\right) = \epsilon(\omega), \omega \in \Omega \qquad (3.11)$$

$$lim_{i\to\infty}\, d\left(\xi_{2n_i}(\omega), \xi_{2m_i-1}(\omega)\right) = \epsilon(\omega), \omega \in \Omega \qquad (3.12)$$

We denote $\Psi = \{\psi: [0,\infty) \to [0,\infty) \text{is continuous, nondecreasing and } \psi^{-1}(\{0\}) = \{0\}\}$, and $\Phi = \{\varphi : [0,\infty) \to [0,\infty) \text{is lower semicontinuous, nondecreasing, and } \varphi^{-1}(\{0\}) = \{0\}\}$.

Now, we state and prove our main result in the following way.

Theorem 3.2 Let $(\Omega,\Sigma,\mu)$is a complete probability measure space,$(X,d,\preceq)$ be a separable complete partially ordered metric space. Let $f,g:\Omega\times X\to X$ be two mappings such that

(a). $f(\omega,.)$ and $g(\omega,.)$ are continuous for all $\omega\in\Omega$;

(b). $f(.,x)$ and $g(.,x)$ are measurable mapping for all $x\in X$;

(c). The pair (f,g) is weakly increasing such thatthere exist $\psi,\varphi\in\Psi$and $\mathcal{F}\in\mathcal{C}$ such that for all comparable $x,y\in X$ and for all $\omega\in\Omega$,we have

$$\psi\left(d\big(f(\omega,x),g(\omega,y)\big)\right)\le\mathcal{F}\left(\psi\big(\mathcal{M}(x,y)\big),\varphi\big(\mathcal{M}(x,y)\big)\right)\qquad(3.13)$$

where

$$\mathcal{M}(x,y,\omega)=\max\left\{d(x,y),d\big(x,f(\omega,x)\big),d\big(y,g(\omega,y)\big),\frac{d\big(x,g(\omega,y)\big)+d\big(y,f(\omega,x)\big)}{2}\right\}$$

Suppose that one of the following two cases is satisfied:

(i). $f$or $g$is continuous;

(ii). $(X,d,\preceq)$is regular.

Then the maps$f$ and g have a common random fixed point.

Proof:Assume that $\xi(\omega),\omega\in\Omega$is a fixed point of f. Taking $x=y=\xi$in (3.13), we have

$$\psi\left(d\left(f(\omega,\xi(\omega)),g(\omega,\xi(\omega))\right)\right)\le\mathcal{F}\left(\psi\left(\mathcal{M}\left(\xi(\omega),\xi(\omega)\right)\right),\varphi\left(\mathcal{M}\left(\xi(\omega),\xi(\omega)\right)\right)\right)\qquad(3.14)$$

where

$$\mathcal{M}(\xi(\omega),\xi(\omega))=\max\left\{d(\xi(\omega),\xi(\omega)),d\left(\xi(\omega),f(\omega,\xi(\omega))\right),d\left(\xi(\omega),g(\omega,\xi(\omega))\right),\frac{d\left(\xi(\omega),g(\omega,\xi(\omega))\right)+d\left(\xi(\omega),f(\omega,\xi(\omega))\right)}{2}\right\}$$

$$=\max\left\{0,0,d\left(\xi(\omega),g(\omega,\xi(\omega))\right),\frac{d\left(\xi(\omega),g(\omega,\xi(\omega))\right)}{2}\right\}$$

$$=d\left(\xi(\omega),g(\omega,\xi(\omega))\right)\qquad(3.15)$$

Hence, from (3.14), we get

$$\psi\left(d\left(\xi(\omega),g(\omega,\xi(\omega))\right)\right)=\psi\left(d\left(f(\omega,\xi(\omega)),g(\omega,\xi(\omega))\right)\right)$$

$$\le\mathcal{F}\left(\psi\left(d\left(\xi(\omega),g(\omega,\xi(\omega))\right)\right),\varphi\left(d\left(\xi(\omega),g(\omega,\xi(\omega))\right)\right)\right)$$

$$\le\psi\left(d\left(\xi(\omega),g(\omega,\xi(\omega))\right)\right)\qquad(3.16)$$

We deduce

$\mathcal{F}\left(\psi\left(d\left(\xi(\omega),g(\omega,\xi(\omega))\right)\right),\varphi\left(d\left(\xi(\omega),g(\omega,\xi(\omega))\right)\right)\right)=\psi\left(d\left(\xi(\omega),g(\omega,\xi(\omega))\right)\right)$. By the property of $\mathcal{F}$, we have

$\psi\left(d\left(\xi(\omega), g\left(\omega, \xi(\omega)\right)\right)\right) = 0$or $\varphi\left(d\left(\xi(\omega), g\left(\omega, \xi(\omega)\right)\right)\right) = 0$.

The functions $\psi$and $\varphi$are in Ψ, so $d\left(\xi(\omega), g\left(\omega, \xi(\omega)\right)\right) = 0$; that is, $\xi(\omega) = g\left(\omega, \xi(\omega)\right)$; that is, $\xi(\omega)$is a common fixed point of $f$and g. Now, if $\xi(\omega)$is a fixed point of$g$, similarly, we get that $\xi(\omega)$is also fixed point of f.

Let the function $\xi_0(\omega): \Omega \to X$be an arbitrary measurable mapping. We can define a sequence ofmeasurable mappings $\{\xi_n(\omega)\}$from Ω to Xas following:

$$\xi_{2n+1}(\omega) = f\left(\omega, \xi_{2n}(\omega)\right),$$

$$\xi_{2n+2}(\omega) = g\left(\omega, \xi_{2n+1}(\omega)\right), \omega \in \Omega, n = 0, 1, 2, \ldots \qquad (3.17)$$

Since the pair (f, g) is weakly increasing mappings, we have

$$\xi_1(\omega) = f\left(\omega, \xi_0(\omega)\right) \preceq g\left(\omega, f\left(\omega, \xi_0(\omega)\right)\right)$$

$$= g\left(\omega, \xi_1(\omega)\right) = \xi_2(\omega),$$

$$\xi_2(\omega) = f\left(\omega, \xi_1(\omega)\right) \preceq g\left(\omega, f\left(\omega, \xi_1(\omega)\right)\right)$$

$$= g\left(\omega, \xi_2(\omega)\right) = \xi_3(\omega),$$

Continuing this process, we get

$$\xi_{2n+1}(\omega) = f\left(\omega, \xi_{2n}(\omega)\right) \preceq g\left(\omega, f\left(\omega, \xi_{2n}(\omega)\right)\right)$$

$$= g\left(\omega, \xi_{2n+1}(\omega)\right) = \xi_{2n+2}(\omega),$$

$$\xi_{2n+2}(\omega) = f\left(\omega, \xi_{2n+1}(\omega)\right) \preceq g\left(\omega, f\left(\omega, \xi_{2n+1}(\omega)\right)\right)$$

$$= g\left(\omega, \xi_{2n+2}(\omega)\right) = \xi_{2n+3}(\omega)$$

Thus for all $n \geq 1$, we have

$$\xi_n(\omega) \preceq \xi_{n+1}(\omega). \qquad (3.18)$$

Without loss of the generality, we can assume that $\xi_n(\omega) \neq \xi_{n+1}(\omega)$ and since $\xi_{2n}(\omega)$ and $\xi_{2n+1}(\omega)$ are comparable, applying (3.13), we have

$$\psi\left(d\left(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)\right)\right) = \psi\left(d\left(f\left(\omega, \xi_{2n}(\omega)\right), g\left(\omega, \xi_{2n+1}(\omega)\right)\right)\right)$$

$$\leq \mathcal{F}\left(\psi\left(\mathcal{M}\left(\xi_{2n}(\omega), \xi_{2n+1}(\omega)\right)\right), \varphi\left(\mathcal{M}\left(\xi_{2n}(\omega), \xi_{2n+1}(\omega)\right)\right)\right) \qquad (3.19)$$

where

$$\mathcal{M}\left(\xi_{2n}(\omega), \xi_{2n+1}(\omega)\right) = \max\Big\{d\left(\xi_{2n}(\omega), \xi_{2n+1}(\omega)\right), d\left(\xi_{2n}(\omega), f\left(\omega, \xi_{2n}(\omega)\right)\right),$$

$$d\left(\xi_{2n+1}(\omega), g\left(\omega, \xi_{2n+1}(\omega)\right)\right), \frac{d\left(\xi_{2n}(\omega), g\left(\omega, \xi_{2n+1}(\omega)\right)\right) + d\left(\xi_{2n+1}(\omega), f\left(\omega, \xi_{2n}(\omega)\right)\right)}{2}\Big\}$$

$$= \max\{d\left(\xi_{2n}(\omega), \xi_{2n+1}(\omega)\right), d\left(\xi_{2n}(\omega), \xi_{2n+1}(\omega)\right),$$

$d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)), \frac{d(\xi_{2n}(\omega), \xi_{2n+2}(\omega)) + d(\xi_{2n+1}(\omega), \xi_{2n+1}(\omega))}{2}\Big\}$

$= \max\left\{d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)), \frac{d(\xi_{2n}(\omega), \xi_{2n+2}(\omega))}{2}\right\}$

$\leq \max\{d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)),$

$\frac{d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) + d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))}{2}\Big\}$

$= \max\{d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)), d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\}$ (3.20)

If $d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) \geq d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$ for some $n \geq 0$, then

$\mathcal{M}(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) = d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))$.

Using (3.19), we have

$\psi\left(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\right)$

$\leq \mathcal{F}\left(\psi\left(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\right), \varphi\left(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\right)\right)$

$\leq \psi\left(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\right)$

Hence

$\mathcal{F}\left(\psi\left(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\right), \varphi\left(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\right)\right) = \psi\left(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\right)$

By the property of$\mathcal{F}$, this implies that $\psi\left(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\right) = 0$or $\varphi\left(d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega))\right) = 0$, which is a contradiction. Therefore, for all $n \geq 0$, $d(\xi_{2n+1}(\omega), \xi_{2n+2}(\omega)) < d(\xi_{2n}(\omega), \xi_{2n+1}(\omega))$. Similarly, we may show that $d(\xi_{2n}(\omega), \xi_{2n+1}(\omega)) < d(\xi_{2n}(\omega), \xi_{2n-1}(\omega))$,for all $n \geq 0$. We deduce that

$d(\xi_{n+1}(\omega), \xi_{n+2}(\omega)) < d(\xi_n(\omega), \xi_{n+1}(\omega)), \forall\, n \geq 0, \omega \in \Omega.$ (3.21)

Hence, the sequence$\{\delta_n(\omega) : \omega \in \Omega\}$given by $\delta_n(\omega) = d(\xi_n(\omega), \xi_{n+1}(\omega))$isa decreasing sequence of non-negative real numbers, there exists $l(\omega) \geq 0$, such that

$\lim_{n\to\infty} \delta_n(\omega) = l(\omega), \omega \in \Omega.$ (3.22)

We claim that $l(\omega) = 0, \omega \in \Omega$. We have

$\lim_{n\to\infty} \mathcal{M}(\xi_n(\omega), \xi_{n+1}(\omega)) = l(\omega)$ (3.23)

Recall that

$\psi\left(d(\xi_{n+1}(\omega), \xi_{n+2}(\omega))\right) \leq \mathcal{F}\left(\psi\left(\mathcal{M}(\xi_n(\omega), \xi_{n+1}(\omega))\right), \varphi\left(\mathcal{M}(\xi_n(\omega), \xi_{n+1}(\omega))\right)\right)$ (3.24)

As$n \to \infty$, by continuity of $\mathcal{F}, \psi$and φ, we get

$\psi(l(\omega)) \leq \mathcal{F}\left(\psi(l(\omega)), \varphi(l(\omega))\right) \leq \psi(l(\omega))$

Using the properties of $\mathcal{F}$, we have $\psi(l(\omega)) = 0$ or $\varphi(l(\omega)) = 0$; that is, $l(\omega) = 0$. We conclude that

$\lim_{n\to\infty} \delta_n(\omega) = 0.$ (3.25)

Now, we will show that $\{\xi_n(\omega):\omega\in\Omega\}$ is a Cauchy sequence, it is sufficient to prove that $\{\xi_{2n}(\omega)\}$ is a Cauchy sequence. We proceed by negation, suppose that $\{\xi_{2n}(\omega)\}$ is not a Cauchy sequence. Since $m_i > n_i$ and $\xi_{2n_i-1}(\omega)$ and $\xi_{2m_i-1}(\omega)$ are comparable, then by (3.13), we get

$$\psi\left(d\left(\xi_{2n_i}(\omega),\xi_{2m_i}(\omega)\right)\right)=\psi\left(d\left(f\left(\omega,\xi_{2n_i-1}(\omega)\right),g\left(\omega,\xi_{2m_i-1}(\omega)\right)\right)\right)$$

$$\leq \mathcal{F}\left(\psi\left(\mathcal{M}\left(\xi_{2n_i-1}(\omega),\xi_{2m_i-1}(\omega)\right)\right),\varphi\left(\mathcal{M}\left(\xi_{2n_i-1}(\omega),\xi_{2m_i-1}(\omega)\right)\right)\right) \quad (3.26)$$

where

$$\mathcal{M}\left(\xi_{2n_i-1}(\omega),\xi_{2m_i-1}(\omega)\right)$$

$$=\max\Big\{d\left(\xi_{2n_i-1}(\omega),\xi_{2m_i-1}(\omega)\right),d\left(\xi_{2n_i-1}(\omega),f\left(\omega,\xi_{2n_i-1}(\omega)\right)\right),$$

$$,d\left(\xi_{2m_i-1}(\omega),g\left(\omega,\xi_{2m_i-1}(\omega)\right)\right),\frac{d\left(\xi_{2n_i-1}(\omega),g\left(\omega,\xi_{2m_i-1}(\omega)\right)\right)+d\left(\xi_{2m_i-1}(\omega),f\left(\omega,\xi_{2n_i-1}(\omega)\right)\right)}{2}\Big\}$$

$$=\max\Big\{d\left(\xi_{2n_i-1}(\omega),\xi_{2m_i-1}(\omega)\right),d\left(\xi_{2n_i-1}(\omega),\xi_{2n_i}(\omega)\right),$$

$$,d\left(\xi_{2m_i-1}(\omega),\xi_{2m_i}(\omega)\right),\frac{d\left(\xi_{2n_i-1}(\omega),\ \xi_{2m_i}(\omega)\right)+d\left(\xi_{2m_i-1}(\omega),\xi_{2n_i}(\omega)\right)}{2}\Big\}$$

By taking the limit as $i\to\infty$, from Lemma 18, we have

$$\lim_{i\to\infty}\mathcal{M}\left(\xi_{2n_i-1}(\omega),\xi_{2m_i-1}(\omega)\right)=max\left\{\epsilon(\omega),0,0,\frac{\epsilon(\omega)+\epsilon(\omega)}{2}\right\}=\epsilon(\omega) \quad (3.27)$$

Hence, from (3.26), we have

$$\psi(\epsilon(\omega))\leq\mathcal{F}\left(\psi(\epsilon(\omega)),\varphi(\epsilon(\omega))\right)\leq\psi(\epsilon(\omega))$$

That is,

$$\mathcal{F}\left(\psi(\epsilon(\omega)),\varphi(\epsilon(\omega))\right)=\psi(\epsilon(\omega))$$

We conclude that $\psi(\epsilon(\omega))=0$ or $\varphi(\epsilon(\omega))=0$, that is $\epsilon(\omega)=0,\omega\in\Omega$, a contradiction, we deduce that $\{\xi_{2n}(\omega)\}$ is a Cauchy sequence in X and so is $\{\xi_n(\omega)\}$, then there exists $\xi(\omega):\Omega\to X$ such that

$$\lim_{n\to\infty}\xi_n(\omega)=\xi(\omega). \quad (3.28)$$

Now, we will distinguish the cases (i) and (ii) of Theorem 3.1.

(i). Without loss of generality, suppose that f is continuous. then

$$\xi(\omega)=\lim_{n\to\infty}\xi_{2n+1}(\omega)$$

$$=\lim_{n\to\infty}f(\omega,\xi_{2n}(\omega))$$

$$=f\left(\omega,\lim_{n\to\infty}\xi_{2n}(\omega)\right)$$

$$=f(\omega,\xi(\omega)).$$

From the beginning of the proof, we get $f(\omega,\xi(\omega)) = \xi(\omega) = g(\omega,\xi(\omega))$.The case that $f$is continuous is treated similarly.

(ii). Now, if the condition (ii) is satisfied. We know that sequence $\{\xi_n(\omega)\}$ is non-decreasing and $\xi_n(\omega) \to \xi(\omega), \omega \in \Omega$ in X;then by regularity of $(X, d, \preceq)$, $\xi_{2n+1}(\omega) \preceq \xi(\omega), \forall n \in \mathbb{N}$. By (3.13)

$$\psi\left(d\left(\xi_{2n+1}(\omega), g(\omega,\xi(\omega))\right)\right) = \psi\left(d\left(f(\omega,\xi_{2n}(\omega)), g(\omega,\xi(\omega))\right)\right)$$

$$\leq \mathcal{F}\left(\psi\left(\mathcal{M}(\xi_{2n}(\omega),\xi(\omega))\right), \varphi\left(\mathcal{M}(\xi_{2n}(\omega),\xi(\omega))\right)\right)$$

where

$$\mathcal{M}(\xi_{2n}(\omega),\xi(\omega)) = \max\left\{d(\xi_{2n}(\omega),\xi(\omega)), d\left(\xi_{2n}(\omega), f(\omega,\xi_{2n}(\omega))\right),\right.$$

$$\left. d\left(\xi(\omega), g(\omega,\xi(\omega))\right), \frac{d\left(\xi_{2n}(\omega), g(\omega,\xi(\omega))\right) + d\left(\xi(\omega), f(\omega,\xi_{2n}(\omega))\right)}{2}\right\}$$

$$= \max\{d(\xi_{2n}(\omega),\xi(\omega)), d(\xi_{2n}(\omega),\xi_{2n+1}(\omega)),$$

$$\left. d\left(\xi(\omega), g(\omega,\xi(\omega))\right), \frac{d\left(\xi_{2n}(\omega), g(\omega,\xi(\omega))\right) + d(\xi(\omega),\xi_{2n+1}(\omega))}{2}\right\}$$

By taking the limit as $n \to \infty$, we have

$$\lim_{n\to\infty} \mathcal{M}(\xi_{2n}(\omega),\xi(\omega)) = d\left(\xi(\omega), g(\omega,\xi(\omega))\right). \qquad (3.29)$$

Thus

$$\psi\left(d\left(\xi(\omega), g(\omega,\xi(\omega))\right)\right) \leq \limsup_{n\to\infty} \psi\left(d\left(\xi_{2n+1}(\omega), g(\omega,\xi(\omega))\right)\right)$$

$$\leq \limsup_{n\to\infty} \mathcal{F}\left(\psi\left(\mathcal{M}(\xi_{2n}(\omega),\xi(\omega))\right), \varphi\left(\mathcal{M}(\xi_{2n}(\omega),\xi(\omega))\right)\right)$$

$$\leq \mathcal{F}\left(\psi\left(d\left(\xi(\omega), g(\omega,\xi(\omega))\right)\right), \varphi\left(d\left(\xi(\omega), g(\omega,\xi(\omega))\right)\right)\right)$$

$$\leq \psi\left(d\left(\xi(\omega), g(\omega,\xi(\omega))\right)\right)$$

Hence

$$\mathcal{F}\left(\psi\left(d\left(\xi(\omega), g(\omega,\xi(\omega))\right)\right), \varphi\left(d\left(\xi(\omega), g(\omega,\xi(\omega))\right)\right)\right) = \psi\left(d\left(\xi(\omega), g(\omega,\xi(\omega))\right)\right)$$

We conclude that $\psi\left(d\left(\xi(\omega), g(\omega,\xi(\omega))\right)\right) = 0$ or $\varphi\left(d\left(\xi(\omega), g(\omega,\xi(\omega))\right)\right) = 0$; that is, $d\left(\xi(\omega), g(\omega,\xi(\omega))\right) = 0$ and so $\xi(\omega) = g(\omega,\xi(\omega))$. From the beginning of the proof, we get $f(\omega,\xi(\omega)) = \xi(\omega) = g(\omega,\xi(\omega))$.

The proof of the theorem is completed.

COMPETING INTERESTS

The authors declare that there is no conflict of interests regarding the publication of this paper.

AUTHOR'S CONTRIBUTION

All authors contributed equally and significantly to writing this paper. All authors read and approved the final paper.

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