RAMANUJAN'S CONTRIBUTION TO MATHEMATICAL WORLD

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ABSTRACT

This review paper provides a glimpse of Ramanujan's contributions in the field of mathematics viz. analysis, number theory, infinite series, continued fractions, partitions, asymptotic expansions and their applications in the field of science and technology. In this paper we discuss briefly the several topics written by Ramanujan in his notebooks

Keywords**: *Asymptotic expansions and approximations, continued fractions, Infinite series, Integrals, Number theory,***

I. INTRODUCTION

Brief history of Srinivasa Ramanujan:- Srinivasa Ramanujan was born on 22 december 1887 in Erode, Tamil Nadu. He was self taught and worked in almost isolation from the mathematical community of his time. He independently rediscovered many existing results and making his own unique contributions, making him a raw genius. He believed his inspiration came from hindu goddess Nmagiri. But he is famous for his unique style, often leaping from insight to insight without formally proving the logical steps in between "His idea as to what constituted, a mathematical proof were of the most shadowy description" said G.H.Hardy, Ramanujan's mentor and one of his few collaborators. He died on 26 april 1920, at the age of 32 and illuminated the path of many mathematicians in the areas of research.

His work was in the area of analysis, number theory, infinite series and continued fractions. The applications of his work are found in different areas of science and technology apart from mathematics.

1) Elementary mathematics:-

Ramanujan's work in the field of high school algebra is well appreciated . He enjoyed finding equal sums of powers, the famous taxi number 1729 is one of the example of this. 1729 is the smallest number discovered by him which can be expressed as sum of two cubes in two different ways.

$$1729 = 1^3 + 12^3 = 9^3 + 10^3$$

There are some more identities given by Ramanujan :-

(a) If $a + b + c = 0$ **then** $2(ab + bc + ca)^4 = a^4(b - c)^4 + b^4(c - a)^4 + c^4(a - b)^4$

(b) Formula in polynomial identity:-

Let $$F_{2m}(a,b,c,d) = (a+b+c)^{2m} + (b+c+d)^{2m} - (c+d+a)^{2m} - (d+a+b)^{2m} + (a-d)^{2m} - (b-c)^{2m}$$

then $$64F_6(a,b,c,d)\,F_{10}(a,b,c,d) = 45F_8{}^2(a,b,c,d).$$

(c) $2\sin\left(\frac{\pi}{18}\right) = \sqrt{2-\sqrt{2+\sqrt{2+\sqrt{2-\cdots}}}}$, **where the sequence of signs -,+,+,... has a period 3.**

2) Number Theory:-

Ramanujan did a lot of work in number theory. Theta function and modular equations discovered by him have applications to number theory. He also worked on partitions, sum of squares, triangular number and gave Ramanujan's Tau function. Hardy-Ramanujan 'circle method' which is useful in number theory can be found in his notebook . It also contained theorem on theory of numbers, reproduced below

Theorem:-

Let a, b, A, B denote positive integers satisfying the conditions (a, b) = 1 = (A, B), ab ≠ square of number. Suppose that every prime p ≡ B(modA) with (p, 2ab) = 1 is expressible in the form $ax^2 - by^2$ for some integers x, y. Then every prime q such that q ≡ -B (mod A) and (q,2ab) = 1 is expressible in the form $bX^2 - aY^2$ for some integers X and Y.

3) Infinite Series:-

a) Ramanujan presented two derivation of 1+2+3+... = -1/12 in ch8 of his first notebook and also discovered most interesting formula for infinite series of π.

$$\frac{1}{\pi} = \frac{2\sqrt{2}}{9801}\sum_{k=0}^{\infty}\frac{(4k)!\,(1103+26390k)}{(k!)^4(396)^{4k}}$$

Ramanujan series for π converges extraordinarily rapidly and forms the basis of some of fastest algorithms currently used to calculate π.

b) Let $\zeta(s)$ denote the Riemann zeta function, if $\alpha,\beta > 0, \alpha\beta = \pi^2$ and n is any non zero integer then

$$\alpha^{-n}\left(\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\alpha k}-1}\right) = (-\beta)^{-n}\left(\frac{1}{2}\zeta(2n+1) + \sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\beta k}-1}\right) - 2^{2n}\sum_{k=0}^{n+1}\frac{\beta_{2k}}{(2k)!}\frac{\beta_{2n+2-2k}}{(2n+2-2k)!}\alpha^{n+1-k}\beta^{k}$$

where $\beta_j, j \geq 0$ denote the jth Bernoulli number*.

Further, if $\alpha = \beta = \pi$ and n is odd and positive number, then $\zeta(2n+1) = 2^{2n}\pi^{2n+1}\sum_{k=0}^{n+1}(-1)^{k+1}\frac{\beta_{2k}}{(2k)!}\frac{\beta_{2n+2-2k}}{(2n+2-2k)!} - 2\sum_{k=1}^{\infty}\frac{k^{-2n-1}}{e^{2\pi k}-1}$

Ramanujan derived many formulas of this type which summed many infinite series in closed form and derived several summation formula for example Abel-Plana summation formula.

* Bernoulli numbers B_n are sequence of signed rational numbers that can be defined by the exponential generating function $\frac{x}{e^x-1} = \sum_{n=0}^{\infty}\frac{B_n x^n}{n!}$

4) Integrals:-

Ramanujan did more efforts to infinite series than to integrals so there are several integrals that bear his name and are commonly used in research now days. Several of his paper focus on integrals. For example

Let $n \geq 0$ put $v = u^n - u^{n-1}$ and define $\Phi(n) = \int_0^1 \frac{\log u}{v}dv$ then for $n > 0$ $\Phi(n) + \Phi\left(\frac{1}{n}\right) = \frac{\pi^2}{6}$ which was proved by Evans and related to dilogarithm $Li_2(s)$ defined for any complex number $'s'$ as

$$Li_2(s) = -\int_o^s \frac{\log(1-u)}{u}du$$

where the principal branch of $\log w$ is taken. Ramanujan studied the dilogrithms, trilogrithms, several functions related to dilogrithm. Ramanujan's most powerful integral theorem is his 'Master Theorem' given as $\int_0^\infty x^{n-1}\sum_{k=0}^{\infty}\frac{\Phi(k)(-x)^k}{k!}dx = \Gamma(n)\Phi(-n)$.

5) Asymptotic expansions and approximations:-

Ramanujan is well known for this asymptotic formula in number theory. In particular, for asymptotic series for the partition function p(n) with Hardy. Partition theory deals with how integers can be broken down into sums. Euler began this theory and grew with collaborations of Ramanujan & G.H.Hardy at Cambridge. Partition of a positive number n is defined to be a sequence of positive integers, whose sum is n. The order of the summands is unimportant in partitions of n, but for consistency, partitions of n will be written with summands in a non increasing order.

Hardy and Ramanujan Asymptotic Partition formula:- For a positive integer n, let p(n) denotes the number of unordered partitions of n, that is, unordered sequences of positive integers which sum to n, then the value of p(n) is given asymptotically by

$$p(n) \sim \frac{1}{4n\sqrt{3}}e^{2\pi\sqrt{\frac{n}{6}}}$$

Here we discuss one example of Ramanujan discovery

As $\mathbf{t \to 0^+}$ $\mathbf{F(t) = 2\sum_{n=0}^{\infty}(-1)^n\left(\frac{1-t}{1+t}\right)^{n(n+1)} \sim 1 + t + t^2 + 2t^3 + 5t^4 + 17t^5 + \cdots}$

More precisely as $t \to 0^{+}$ $\quad F(t) \sim \left(\frac{1+t}{1-t}\right)^{\frac{1}{4}} \sum_{n=0}^{\infty} \frac{(-1)^n E_{2n}}{2^{2n} n!} \left(\log\left(\frac{1+t}{1-t}\right)\right)^n$

Obviously latter one is obtained from the first but latter was not given by Ramanujan**.**

6) Continued Fraction:-

The Rogers Ramanujan continued fraction

$$R(q) = \frac{q^{\frac{1}{5}}}{1+}\,\frac{q}{1+}\,\frac{q^2}{1+}\,\frac{q^3}{1+\cdots} = q^{\frac{1}{5}} \prod_{n=1}^{\infty} \frac{(1-q^{5n-1})(1-q^{5n-4})}{(1-q^{5n-2})(1-q^{5n-3})} = \frac{q^{\frac{1}{5}}(q;q^5)_\infty (q^4;q^5)_\infty}{(q^2;q^5)_\infty (q^3;q^5)_\infty} \quad |q| < 1$$

In his notebook, there were almost 2000 contributions to continued fractions and no one in the history of mathematics had such skill of finding the continued fractions of various functions like that of Ramanujan. He also claimed that $R(e^{-8\pi}) = \sqrt{c^2+1} - c$ where $2c = 1 + \frac{a+b}{a-b}\sqrt{5}\ ; a = 3+\sqrt{2}-\sqrt{5},\ b = (20)^{1/4}$

Ramanujan also examines R(q) for $|q| > 1$ when q is primitive nth root of unity. He also asserted that if $q = e^{2\pi i m/n}$, where (m,n)=1 then R(q) diverges if n is multiple of 5 and otherwise converges. He also discovered continued fraction for several products of gamma functions. For example if x, m, n are complex and either m or n is integer or if Re x > 0 then

$$\begin{aligned}
&\left\{\Gamma\left(\frac{1}{2}(x+m+n+1)\right)\Gamma\left(\frac{1}{2}(x-m-n+1)\right)\right. \\
&\qquad \left. - \Gamma\left(\frac{1}{2}(x+m-n+1)\right)\Gamma\left(\frac{1}{2}(x-m+n+1)\right)\right\} \\
&\qquad \div \left\{\Gamma\left(\frac{1}{2}(x+m+n+1)\right)\Gamma\left(\frac{1}{2}(x-m-n+1)\right)\right. \\
&\qquad \left. + \Gamma\left(\frac{1}{2}(x+m-n+1)\right)\Gamma\left(\frac{1}{2}(x-m+n+1)\right)\right\} \\
&\qquad = \frac{mn}{x+}\,\frac{(m^2-1^2)(n^2-1^2)}{3x+}\,\frac{(m^2-2^2)(n^2-2^2)}{5x+\cdots}
\end{aligned}$$

And more interesting application of this formula is continued fraction of π.

$$\pi = \frac{4}{1+}\,\frac{1^2}{2+}\,\frac{3^2}{2+}\,\frac{5^2}{2+\cdots}$$

Applications of Ramanujan's discoveries in science and technology

- The mathematical contributions of Ramanujan have also been widely used in solving various problems in higher scientific fields of specialisation. The diverse specialised higher scientific fields include the particle physics, statistical mechanics, computer science, space science, cryptology, polymer chemistry, medical science. Ramanujan's mathematical methods are being used in designing better blast furnaces for smelting metals and splicing telephone cables for communication, as well.
- The most celebrated applications of the' Ramanujan conjecture' is explicit construction of Ramanujan graphs by Lubotzky, Philips and Sarnak. The connection of this conjecture with other conjectures of A.weil in algebraic geometry opened up new areas of research.
- Ramanujan developed exceptionally efficient ways of calculating pi that were latter incorporated into computer algorithms, which is also used in various formulae of physics and engineering to describe such periodic phenomenon as the motion of pendulums, the vibration of strings and alternating electric currents.
- Ramanujan's Master Theorem is a technique that provides an analytic expression for the Mellin transform of an analytic function. Ramanujan widely used it to calculate definite integrals and infinite series. Higher dimensional versions of this theorem also appear in quantum-physics.
- Ramanujan's different identities are used in simple physical models and 2-D models.
- Ramanujan Modular functions provides symmetry which gives evidence about possibility of multiple dimensions of universe i.e. 10 or more.
- Ramanujan ideas of number theory has the best application in Cryptography.
- A new formula, inspired by the mysterious work of Srinivasa Ramanujan could improve our understanding of black holes. Devised by Ken Ono of Emory university in Atlanta, Georgia the formula concerns a type of function called Mock Modular form. These functions are now used to complete the entropy of black holes. This property is linked to the startling prediction by Stephan Hawking that black holes emit radiation.

Ramanujan independently compiled nearly 3,900 results (mostly identities and equations), during his short life. Approximately all his claims have now been proven correct. His original and highly unconventional results for example prime and the Ramanujan theta function, have inspired a vast amount of further research. *The* Ramanujan *Journal*, a peer-reviewed scientific journal, was established by springer to publish work in all areas of mathematics influenced by Ramanujan.

Ramanujan was deeply religious and gave all credit of his substantial mathematical capacities to divinity: "An equation for me has no meaning," he once said, "unless it expresses a thought of God."

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